Chapter 3

TOOLS TO ANALYSE MOTIONS

1. *INTRODUCTION*

In Chapter 2, the mathematical model of the p-version of the FEM has been derived including geometrical non-linearity for thick, asymmetric composite laminated rectangular plates. In order to solve the non-linear equations of motion, the Newmark method was presented. In this chapter different tools that can be used to characterize the responses of non-linear systems are presented.

In non-linear vibrations, there are several parameters that influence the time dependence of the response: time variation, space dependence and amplitude of the external excitation, properties of the structure, initial and boundary conditions, etc. Depending on these parameters, the oscillations may be periodic – including harmonic, sub-harmonic and super-harmonic – quasi-periodic or even chaotic [3.1, 3.7].

Unlike equilibrium equations, periodic solutions are characterized by one basic frequency ω. The spectrum of a periodic signal consists of a spike at the frequency 0 and spikes at integer multiples of ω.

A quasi-periodic solution is a dynamic solution characterized by two or more incommensurate[1] frequencies. Although the waveform of a quasi-periodic signal may look complex because of the presence of many sinusoids in it, calculating its spectrum would reveal its simplicity. In principle, the spectrum can be used to distinguish a quasi-periodic signal from a periodic signal in that the spikes in the spectrum of a quasi-periodic signal are not spaced at integer multiples of a particular

[1] Two frequencies ω_1 and ω_2 are said to be incommensurate if ω_1 / ω_2 is an irrational number.

frequency [3.2]. Poincaré Maps are also used to determine the stability of a quasi-periodic solution.

Chaotic solutions will also be studied in this thesis; there is no precise definition for a chaotic solution because it cannot be represented through standard mathematical functions. A chaotic solution is an aperiodic solution which is endowed with some special characteristics. From a practical point of view, chaos can be defined as a bounded steady-state behaviour that is not an equilibrium solution or a periodic solution or a quasi-periodic solution. The attractor associated with a chaotic motion in state space is not a simple geometrical object like a finite number of points, a closed curve or a torus. Chaotic attractors are complicated geometrical objects that possess fractal dimensions. The spectrum of a chaotic signal contains spikes that indicate the predominant frequencies of the signal [3.2].

In this chapter, tools such as Fourier spectra [3.2], Poincaré Maps [3.3, 3.5] and Lyapunov exponents [3.3, 3.5], which are used to characterize the responses of non-linear systems, will be presented.

2. *FOURIER SPECTRA*

The Fourier or frequency spectra help in distinguishing among periodic, quasi-periodic, and chaotic motions and are typically used to study stationary signals. The frequency spectrum can be either amplitude or a power spectrum. In an amplitude spectrum, the Fourier amplitude is displayed at each frequency. On the other hand, in a power spectrum, the square of the Fourier amplitude per unit time is displayed at each frequency [3.2].

The Fourier transform of a signal $x(t)$ is defined as

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-2i\pi\omega t} dt \tag{3.4}$$

where ω denotes the frequency, $X(\omega)$ is an integrable[2] complex quantity.

In theory, the Fourier transform can be used to determine the frequency content of a signal $X(\omega)$ if it is known for $-\infty < \omega < +\infty$ and is integrable. However, a stationary signal that exists for all ω is not integrable. Besides, in practice, $X(\omega)$ is known for only a finite length of time T_c and hence the so called finite Fourier transform is used:

$$X(\omega, T_c) = \int_0^{T_c} x(t) e^{-2i\pi\omega t} dt \quad (3.5)$$

where, again, $X(\omega, T_c)$ is a complex quantity.

Fourier methods will only be employed to determine the harmonic content of periodic oscillations. The finite Fourier transform provides a mechanism for representing a signal as the sum of simple sine and cosine functions, which correspond to discrete lines in the frequency spectrum. In this work, the signal is a time series obtained from the numerical integration of the equations of motion. This time series is collected over a finite time T_c and consists of a discrete number of points obtained at a chosen sampling frequency. The period of the stationary motions is related with the excitation period, and one easily selects a length of numerical data that coincides with the period. These data is modelled as a sum of sine and cosine functions of time t [3.2]. To obtain the frequency spectra of quasi-periodic and chaotic oscillations, other tools of signal processing, like the power spectral density function are recommended.

The Fourier transform of discrete data is obtained using the discrete Fourier transform (DFT). It is a procedure for modifying the Fourier transform so as to permit its computation on a digital computer. Hence, a discrete Fourier transform is an approximation of the continuous Fourier transform. A special case of the DFT transform is the fast Fourier transform (FFT) It is essentially an efficient computational scheme that takes advantage of certain symmetry properties in the

[2] $X(\omega)$ is integrable if $\int_{-\infty}^{+\infty} |x(t)| dt < +\infty$

cosine and sine functions at their points of evaluation in order to achieve speed over conventional methods [3.2].

Suppose that we sample a periodic, continuous time function $X(t)$ at a sequence of N points with equal time intervals of length Δ seconds, starting at time t_0, with period $N\Delta$; the discrete Fourier transform is given by

$$X(t)=\frac{1}{N}\left[c_1+2\sum_{n=2}^{(N+1)/2}c_{2n-2}cos\left[\frac{2\pi(n-1)(t-t_0)}{N\Delta}\right]-\right.$$
$$\left.-2\sum_{n=2}^{(N+1)/2}c_{2n-1}sin\left[\frac{2\pi(n-1)(t-t_0)}{N\Delta}\right]\right] \tag{3.6}$$

and the coefficients c_k, $k=1,...,N/2$ are given by

$$\begin{aligned} c_1 &= \sum_{n=1}^{N} s_n \\ c_{2m-1} &= -\sum_{n=1}^{N} s_n sin\left[\frac{2\pi(m-1)(n-1)}{N}\right], \quad m=2,...,N/2 \\ c_{2m-2} &= \sum_{n=1}^{N} s_n cos\left[\frac{2\pi(m-1)(n-1)}{N}\right], \quad m=2,...,(N/2)+1 \end{aligned} \tag{3.7}$$

If N is odd, c_m is defined from 2 to $\frac{N+1}{2}$. The dominant frequencies correspond to the higher values of P_k, where P_k are the components of a vector $\{P\}$ of length $N/2$ as follows

$$\begin{aligned} P_1 &= |c_1| \\ P_k &= \sqrt{c_{2k-2}^2+c_{2k-1}^2} \quad k=2,...,(N+1)/2 \end{aligned} \tag{3.8}$$

These values correspond to the energy spectrum of the signal. In particular, P_k corresponds to the energy level at frequency $\frac{k-1}{N\Delta}$, $k=1,2,...,\frac{N+1}{2}$.

The time series of a periodic motion has the appearance of a uniform trace, and the corresponding spectrum has one basic frequency. If the response spectrum of a system excited by a harmonic excitation contains solely a line at the excitation frequency, the motion is a linear periodic motion. On the other hand, if the response spectrum contains lines at frequencies which are multiples of the excitation frequency, then the motion is non-linear periodic. The spectrum of a periodic motion consists of a single basic frequency. When the spectrum has *n* basic frequencies (i.e., *n* incommensurate frequencies), the corresponding motion is no longer periodic and is called an *n*-period quasi-periodic motion. The spectrum of a chaotic motion has a continuous or broadband character [3.5].

The spectra of random motions such as noise also have continuous or broadband character, but chaotic motions can be distinguished from noise by using the character of the spectrum and tools, such as dimension calculations and Lyapunov exponents. For the spectrum associated with a chaotic motion, the Fourier amplitudes are frequency dependent in the broadband region. These amplitudes scale as $1/\omega^{\alpha}$, where ω is the frequency and α is a positive integer. For the spectrum associated with a random motion, the Fourier amplitudes in the broadband region are either frequency independent or frequency dependent and do not follow the $1/\omega^{\alpha}$ scaling law [3.2].

The Fourier analysis is not well suited for signals with transient effects that occur over a short period of time because it is not localized in time. This problem can partly be overcome by conducting Fourier analysis in different time windows. The location of the time window adds a time dimension to the overall analysis. For a signal with short-lived transient events, it is desirable to use functions such as wavelets, that are localized in time and frequency to represent the signal rather than sine and cosine functions that extend over all time [3.2].

3. *POINCARÉ MAPS*

In this section, we consider periodic solutions of dynamical systems, especially continuous-time systems. For continuous values of time, the evolution of a system is governed by either an autonomous or a non-autonomous system of differential equations. Only non-autonomous systems are studied. In these systems the equations are of the form

$$\{\dot{x}\} = \{F(\{x\}, t)\} \tag{3.9}$$

Where $\{x\}$ is finite dimensional, $\{x\} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\{F\}$ explicitly depends on *t*. The vector $\{F\}$ is often referred to as vector field, the vector $\{x\}$ is called state vector because it describes the state of the system, and the space $\mathbb{R}^n$ in which $\{x\}$ evolves is called a state space. A state space is called a phase space when one-half of the states are displacements and the other one-half are velocities [3.2].

Let the initial state of the system at time t_0 be $\{x_0\}$, and let *I* represent a time interval that includes t_0. In general, a projection of a solution $\{x(t, t_0, x_0)\}$ of (3.9) onto the *n*-dimensional state space is referred to as trajectory or an orbit of the system trough the point $\{x\} = \{x_0\}$. In other words, the solution could be thought of as a point that moves along a trajectory, occupying different positions at different times similar to the way a planet moves through the space. An orbit is represented by $\gamma(\{x_0\})$ or Γ. The orbit obtained for times $t \geq 0$ passing through the point $\{x_0\}$ at $t = 0$ is called a positive orbit and is denoted by $\gamma^+(\{x_0\})$; the orbit obtained for times $t \leq 0$ is called a negative orbit and is denoted by $\gamma^-(\{x_0\})$. Also, $\Gamma = \gamma^+(\{x_0\}) \cup \gamma^-(\{x_0\})$, where the symbol $\cup$ stands for the union operator.

Equation (3.9) is also referred to as an evolving equation. Let the evolution of the system described by this equation be controlled by a set of parameters *E*. To make this parameter dependence explicit, we describe the evolution by

$$\{\dot{x}\} = \{F(\{x\}, t; E)\} \tag{3.10}$$

where *E* is a vector of control parameters. Formally, $E \in \mathbb{R}^n$ and the vector function $\{F\}$ can be represented as $\{F\} : \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Next, some facts from the theory of ordinary-differential equations are stated [3.4]. If the scalar components of $\{F\}$ are C^0 in a domain D of the $(\{x\}, t)$ space, then a solution $\{x(t, t_0, x_0)\}$ satisfying the conditions $\{x\} = \{x_0\}$ at $t = t_0$ exists in a small time interval around t_0 in D. Moreover, if the scalar components of $\{F\}$ are C^1 in D, then the solution $\{x(t,t_0,\{x_0\})\}$ is unique in a small time interval around t_0. If the existence and uniqueness of solutions of a system of the form (3.10) are ensured, then this system is deterministic. This means that two integral curves starting from two different initial conditions cannot intersect each other in the extended state space.

Equation (2.71) depends explicitly on time; therefore it represents a non-autonomous system.

A dynamic solution $\{x\} = \{X(t)\}$ of a continuous time system is periodic with least period T if $\{X(t+h)\} = \{X(t)\}$ and $\{X(t+\tau)\} \neq \{X(t)\}$ for $0 < \tau < T$. For these solutions, Poincaré introduced the notion of orbital stability.

Let Γ_1 represent the orbit of *u* and Γ_2 represent the orbit of *v* for all times. The periodic solutions *u* and *v* have different periods T_1 and T_2, and hence, the corresponding motions evolve on different time scales. The orbit Γ_1 is said to be orbitally stable if, given a small number $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|u(t=0) - v(t=\tau)\| < \delta$ for some τ, then there exist t_1 and t_2 for which $\|u(t_1) - v(t_2)\| < \varepsilon$. Further, if Γ_2 tends to Γ_1 as $t \to \infty$, then we say Γ_1 is asymptotically stable. In Poincaré stability, we examine how "close" orbits are in the state space.

In the following chapter Poincaré maps are used to determine if the solutions are periodic, quasi-periodic or chaotic [3.3].

A Poincaré section is a hypersurface[3] in the state space that is transverse to the flow of a given system of equations. The state space is given by

$$\{nx(t)\}.\{F(\{x\};t)\} \neq 0 \tag{3.11}$$

[3] In a *n*-dimensional space, a hypersurface is a surface whose dimension is less than *n*.

where $\{nx(t)\}$ is a vector normal to the selection located at $\{x\}$, $\{F(x;t)\}$ is the vector field describing the flow, and the dot indicates the dot product. A Poincaré section is denoted by Σ. Although the time interval between two successive intersections of a trajectory with a chosen Poincaré section is not constant in all situations, we can collect the points on the Poincaré section by stroboscopically monitoring the state variables at intervals of the period T [3.5].

In Poincaré Maps, a finite number of points corresponds to a periodic motion, an infinite number of points filling up a closed curve corresponds to quasi-periodic motion, and an infinite number of orderly distributed points (usually) corresponds to chaotic motion.

4. *LYAPUNOV EXPONENTS*

For a dynamical system, sensitivity to initial conditions is quantified by the Lyapunov exponents. For example, consider two trajectories with nearby initial conditions on an attractor. When the attractor is chaotic, the trajectories diverge, on average, at an exponential rate characterized by the largest Lyapunov exponent [3.6]. Considering the non-autonomous system given in (3.9), a particular trajectory $\{\tilde{x}_1(t)\}$ and a neighbour of that trajectory (defined by the diameter, of the sphere) at some instance of time t_0, the purpose is to evaluate how another trajectory $\{\tilde{x}_2(t)\}$ diverges from $\{\tilde{x}_1(t)\}$, as the system evolves. This way, the Lyapunov exponent evaluates the time evolution of a sphere's axes.

The variation of the diameter may be expressed as $d(t) = d_0 2^{\lambda t}$ [3.5]. Therefore,

$$\lambda = \frac{1}{t}\log_2\left(\frac{d(t)}{d(0)}\right)$$

If the exponent λ is negative or equal to zero, then the trajectory $\{\tilde{x}_2(t)\}$ does not diverge from $\{\tilde{x}_1(t)\}$; on the other hand, if λ is positive, the trajectory $\{\tilde{x}_2(t)\}$ diverges exponentially from the original orbit, characterizing chaos [3.5].

This concept is also generalized for the spectrum of Lyapunov exponents, $\lambda_i, i = 1, 2, ..., n$, by considering a small n-dimensional sphere of initial conditions, where n is the number of equations (or, equivalently, the number of state variables) used to describe the system. As time progresses, the sphere evolves into an ellipsoid whose principal axes expand (or contract) at rates given by the Lyapunov exponents [3.6]:

$$\lambda_i = \lim_{t\to\infty}\left[\frac{1}{t}\log_2\left(\frac{d_i(t)}{d_i(0)}\right)\right],\ i = 1,..,n$$

where $d_i(t)$ measures the growth of an infinitesimal *n*-sphere of initial conditions at $t = 0$ in terms of the i'th ellipsoidal axis.

The presence of a positive exponent is sufficient to diagnose chaos and represents local instability in a particular direction. If more than one Lyapunov exponent is positive then there is hipercaos. Note that for the existence of an attractor, the overall dynamics must be dissipative, i.e., globally stable, and the total rate of contraction must outweigh the total rate of expansion. Thus, even when there are several positive Lyapunov exponents, the sum across the entire spectrum is negative.

Wolf et al [3.6] explain the Lyapunov spectrum by providing the following geometrical interpretation. First, arrange the *n* principal axes of the ellipsoid in the order of most rapidly expanding to most rapidly contracting. It follows that the associated Lyapunov exponents will be arranged such that

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$$

where λ_1 and λ_n correspond to the most rapidly expanding and contracting principal axes, respectively. Next, recognize that the length of the first principal axis is proportional to $2^{\lambda_1 t}$; the area determined by the first two principal axes is proportional to $2^{(\lambda_1+\lambda_2)t}$; and the volume determined by the first *k* principal axes is proportional to $2^{(\lambda_1+\lambda_2+...+\lambda_k)t}$. Thus, the Lyapunov spectrum can be defined such that the exponential growth of a *k*-volume element is given by the sum of the *k* largest Lyapunov exponents. Note that information created by the system is represented as a change in the volume defined by the expanding principal axes.

Next, the method used by Wolf et *al*. to compute the largest non-negative Lyapunov exponent is presented. It is based in evaluating distances between points and the trajectory. Given the time series $\{x(t)\}$, an n-dimensional phase portrait is reconstructed with delay coordinates [3.6], i.e., a point on the attractor is given by

$$\{\{x(t)\},\{x(t+\tau)\},...,\{x(t+(n-1)\tau)\}\},$$

where τ is the almost arbitrary delay time. Considering $\{Z(t_0)\}$ the nearest neighbour to the initial point $\{x(t_0)\}$ and L_0, the distance between $\{x(t_0)\}$ and $\{Z(t_0)\}$ is given by $L_0 = \|\{x(t_0)\}-\{Z(t_0)\}\|$ where $\| \ \|$ represents the Euclidean norm.

Defining an hipersphere with ray ε centred in $\{x(t_0)\}$ such that $\{Z(t_0)\}$ is inside the hipersphere, i.e., $L_0 = \|\{x(t_0)\}-\{Z(t_0)\}\| < \varepsilon$ the time evolution is followed from $\{x(t_0)\}$ to $\{Z(t_0)\}$ until, in an instant $t_1 = t_0 + \tau$ the distance between those points, L_0', is greater than ε. In that moment, $\{Z(t_0)\}$ is replaced by another neighbour, closer to $\{x(t_1)\}$, that is in the direction of the segment L_0' and such that

$$L_1 = \|\{x(t_1)\}-\{Z(t_1)\}\| < \varepsilon .$$

The process follows until all the points $\{x(t_i)\}$ are analysed. The largest positive Lyapunov exponent is obtained as an average of $\log_2\left(L_i^{'}/L_i\right)$ along the trajectory, i.e.,

$$\lambda_1 = \frac{1}{t_M - t_0}\sum_{i=0}^{M-1}\log_2\left(\frac{L_i^{'}}{L_i}\right)$$

where M is the number of times a new neighbour was chosen close to the trajectory.
In practice, where a finite number of points in the time series are considered and the presence of noise is usual, the selection of a neighbour point placed in the direction of the segment $L_{i-1}^{'}$ is not possible. The criterion is the selection of a point inside a cone of height ε with an angle of $\theta = \pi/9$ and the symmetry axes matches the segment $L_{i-1}^{'}$. If no point is found, the angle is increased. Finally, the closest neighbour is chosen, regardless of θ and ε.

5. *CLOSING COMMENTS*

In this chapter three methods to determine the type of time domain solution were presented.
For the Fourier Spectra of a signal, periodic motion always shows up a discrete frequency spetrum. So does quasi-periodic motion, displaying the two or more incommensurate frequencies involved, and possibly sub-harmonics, higher harmonics, and linear combinations of these. Chaotic motion produces a continuous broadband spectrum with spikes at the dominating frequencies.
Representing motion in a Poincaré map, it is usually easy to distinguish periodic and non-periodic motion. Summing up the possible sets of Poincaré maps, a finite number of points corresponds to a periodic motion, an infinite number of points filling up a closed curve corresponds to quasi-periodic motion, and an infinite number of orderly distributed points (usually) corresponds to chaotic motion.

Finally, Lyapunov exponents essentially measure the average rates of convergence or divergence of nearby orbits in the phase space. A positive Lyapunov exponent is among the strongest indicators of chaotic motion. If $\lambda_i < 0,\ i = 1, n$ then the solution is an equilibrium point; if $\lambda_1 = 0$ and $\lambda_i < 0,\ i = 2, n$, then a periodic solution is obtained; if $\lambda_1 = \lambda_2 = 0$ and $\lambda_i < 0,\ i = 3, n$ then a quasi-periodic solution is obtained.