

Chapter 2

MATHEMATICAL MODEL

1. INTRODUCTION

In this chapter, a hierarchical finite element model for geometrically non-linear vibration in unsymmetrical laminated plates made of composite materials is developed. The model will be presented in the time domain.

2. MATHEMATICAL MODEL

2.1 - Field Equations

In this section, a hierarchical finite element method for asymmetric composite plates is presented. The plate with constant thickness h , width a and length b , is composed of orthotropic layers oriented at different angles θ .

The origin of the co-ordinate system is located at the middle plane with the z -axis being normal to the mid-plane. Using the first order shear deformation theory (FSDT), Kirchhoff's hypothesis is relaxed by removing the third part, i.e., the transverse normals do not remain perpendicular to the midsurface after deformation. The inextensibility of transverse normals requires that w is not a function of the thickness coordinate, z . The displacement field (Figure 1) of the first-order theory is of the form [2.1]

$$u(x, y, z, t) = u^0(x, y, t) + z\Phi_y(x, y, t) \quad (2.1)$$

$$v(x, y, z, t) = v^0(x, y, t) - z\Phi_x(x, y, t) \quad (2.2)$$

$$w(x, y, z, t) = w^0(x, y, t) \quad (2.3)$$

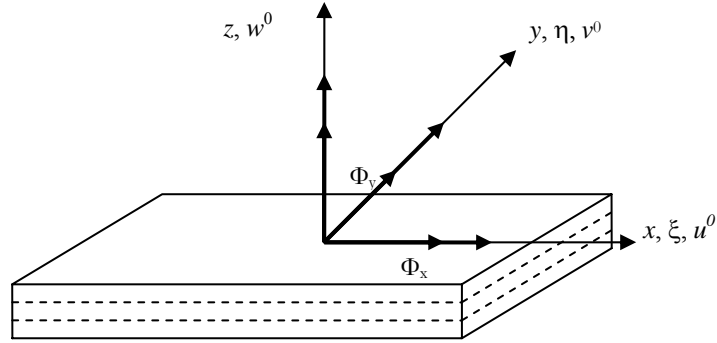


Figure 1 - Plate Element, coordinates, mid-plane displacements and rotations

where (u^0, v^0, w^0) are the displacements along the coordinate lines of a material point on the xy plane, therefore in the mid-plane ($z=0$), and where Φ_x and Φ_y are

$$\frac{\partial u}{\partial z} = \Phi_y, \quad \frac{\partial v}{\partial z} = -\Phi_x$$

the rotations of a transverse normal about the x -axis and the y -axis, respectively. The functions $(u^0, v^0, w^0, \Phi_x, \Phi_y)$ are unknown and are to be determined. For thin plates, i.e., when the plate in-plane characteristic dimension to thickness ratio is on the order 50 or higher ($a/h \geq 50$) [2.2], the rotation functions Φ_x and Φ_y should approach the respective slopes of the transverse deflection:

$$\Phi_x = \frac{\partial w_0}{\partial y}, \quad \Phi_y = -\frac{\partial w_0}{\partial x}$$

In this case, the first order shear deformation theory becomes identical to the classical plate theory where Kirchhoff's hypothesis is followed [2.5].

The non-linear strains associated with the displacement field in Figure 1 are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{zx} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \gamma_{zx}^0 \\ \gamma_{yz}^0 \end{Bmatrix} - \begin{Bmatrix} z\kappa_x \\ z\kappa_y \\ z\kappa_{xy} \\ 0 \\ 0 \end{Bmatrix} \quad (2.4)$$

where ε_x^0 , ε_y^0 and γ_{xy}^0 are the in-plane strain components at $z = 0$ defined by the von Kármán non-linear strain-displacement relationships[2.3]:

$$\varepsilon_x^0 = u_{,x}^0 + \frac{1}{2}(w_{,x}^0)^2, \varepsilon_y^0 = v_{,y}^0 + \frac{1}{2}(w_{,y}^0)^2, \gamma_{xy}^0 = u_{,y}^0 + v_{,x}^0 + w_{,x}^0 w_{,y}^0 \quad (2.5)$$

where $u_{,x}^0$ represents the partial derivate $\frac{\partial u_0}{\partial x}$.

Figure 2 shows these displacements.

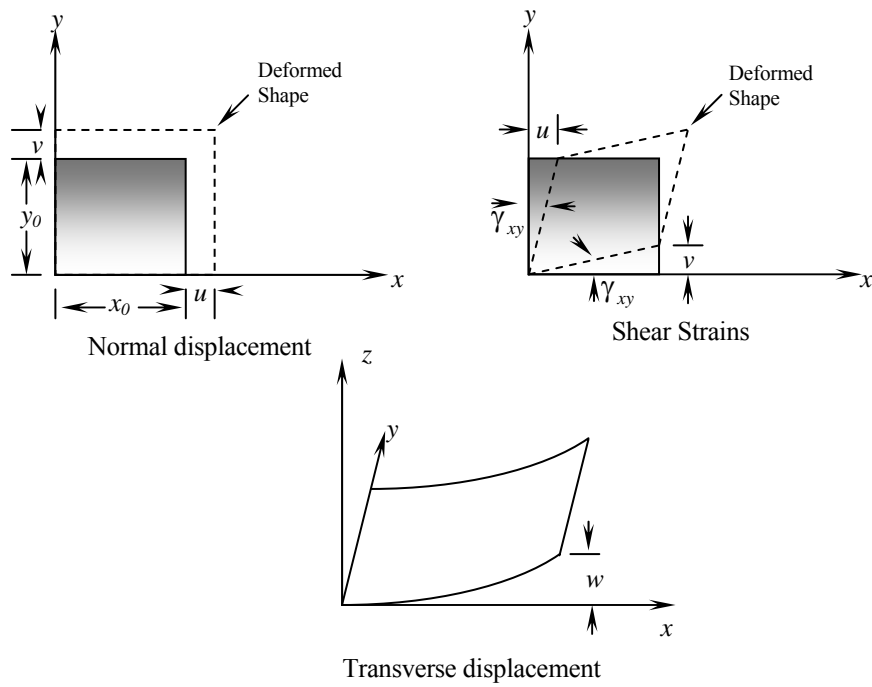


Figure 2 – Displacements of a plate

The terms κ_x , κ_y and κ_{xy} in equation (2.4) are the curvatures or bending strains, which are given by:

$$\kappa_x = -\frac{\partial \Phi_y^0}{\partial x}, \kappa_y = \frac{\partial \Phi_x^0}{\partial y} \text{ and } \kappa_{xy} = -\frac{\partial \Phi_y^0}{\partial y} + \frac{\partial \Phi_x^0}{\partial x} \quad (2.6)$$

The transverse shear strains are

$$\gamma_{zx}^0 = w_{,x}^0 + \Phi_{,y}^0, \gamma_{yz}^0 = w_{,y}^0 - \Phi_{,x}^0 \quad (2.7)$$

For each element, the middle plane in-plane displacements and the rotations are expressed in the form:

$$\begin{Bmatrix} u^0 \\ v^0 \\ w^0 \\ \Phi_y^0 \\ \Phi_x^0 \end{Bmatrix} = [N] \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \quad (2.8)$$

where $\{q_u\}$, $\{q_v\}$ and $\{q_w\}$ are the vectors of generalised in and out-of-plane displacements, and $\{q_{\Phi_y}\}$ and $\{q_{\Phi_x}\}$ are the vectors of generalised rotations. The complete matrix of shape functions

$$[N] = \begin{bmatrix} \{N^u\}^T & 0 & 0 & 0 & 0 \\ 0 & \{N^v\}^T & 0 & 0 & 0 \\ 0 & 0 & \{N^w\}^T & 0 & 0 \\ 0 & 0 & 0 & \{N^{\Phi_y}\}^T & 0 \\ 0 & 0 & 0 & 0 & \{N^{\Phi_x}\}^T \end{bmatrix} \quad (2.9)$$

is constituted by the row vectors of bi-dimensional in-plane, out-of-plane and rotational shape functions, which are, respectively,

$$\{N^u\}^T = \{g_1(\xi)g_1(\eta), g_1(\xi)g_2(\eta), \dots, g_{p_i}(\xi)g_{p_i}(\eta)\} \quad (2.10)$$

$$\{N^v\}^T = \{f_1(\xi)f_1(\eta), f_1(\xi)f_2(\eta), \dots, f_{p_o}(\xi)f_{p_o}(\eta)\} \quad (2.11)$$

$$\{N^{\Phi_y}\}^T = \{\Phi_{y_1}(\xi)\Phi_{y_1}(\eta), \Phi_{y_1}(\xi)\Phi_{y_2}(\eta), \dots, \Phi_{y_{p_{\Phi_y}}}(\xi)\Phi_{y_{p_{\Phi_y}}}(\eta)\} \quad (2.12)$$

$$\{N^{\Phi_x}\}^T = \{\Phi_{x_1}(\xi)\Phi_{x_1}(\eta), \Phi_{x_1}(\xi)\Phi_{x_2}(\eta), \dots, \Phi_{x_{p_{\Phi_x}}}(\xi)\Phi_{x_{p_{\Phi_x}}}(\eta)\} \quad (2.13)$$

The vectors $\{g\}$, $\{f\}$, $\{\Phi_y\}$ and $\{\Phi_x\}$ are the vectors of in-plane, transverse, and rotational one dimensional displacement shape functions; p_o , p_i , p_{Φ_y} and p_{Φ_x} are the

numbers of respective transverse, middle plane, rotation about y and rotation about x , displacement shape functions employed; ξ and η are the local coordinates, which are given by:

$$\xi = 2x/a, \quad \eta = 2y/b \quad (2.14)$$

In the hierarchical finite element method, one is free to choose the number and set of displacement shape functions to be applied in the definition of the element. Increasing the order of the shape functions that represent the displacements within the element increases the accuracy of the element. In a plate element, there is a set of shape functions for the transverse displacements, a set of shape functions for the in-plane displacements and one set of shape functions for each rotation.

An element of a layer is now considered and the plate geometric axes are x and y , as in Figure 3. The principal material axes are labelled 1 and 2, that is, the 1 direction is parallel to the fibbers and the 2 direction is normal to them.

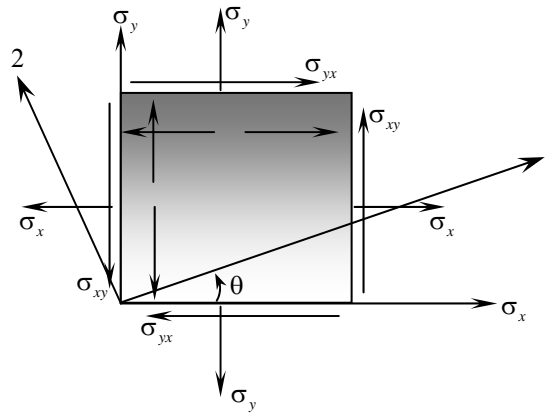


Figure 3 – Laminae Coordinate System

Mohr's circle analysis in basic strength of materials can be applied to establish that

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [T]_{CL} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \quad (2.15)$$

Analogously, the strain relationship is given by

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = [T]_{CL} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}, \quad (2.16)$$

where

$$[T]_{CL} = \begin{bmatrix} m^2 & n^2 & +2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & (m^2 - n^2) \end{bmatrix} \quad (2.17)$$

where $m = \cos(\theta)$, $n = \sin(\theta)$, and θ is the positive angle defined, and CL refer to the 1-2 plane only. The effects of transverse shear deformation are shown in the inclusion of the relations $\sigma_4 - \varepsilon_4$ and $\sigma_5 - \varepsilon_5$ in composite materials [2.4], therefore the equations (2.15)-(2.17) are modified to:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} \quad (2.18)$$

Where

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & -2mn \\ 0 & 0 & m & -n & 0 \\ 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & (m^2 - n^2) \end{bmatrix} \quad (2.19)$$

Multiplying (2.18) by $[T]^{-1}$, the relationships can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (2.20)$$

The $[T]^{-1}$ matrix is obtained by replacing θ by $-\theta$ in $[T]$, and is given by

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 2mn \\ 0 & 0 & m & n & 0 \\ 0 & 0 & -n & m & 0 \\ mn & -mn & 0 & 0 & (m^2 - n^2) \end{bmatrix} \quad (2.21)$$

Using Hooke's law that relates stresses and strains, the general equations for a lamina of composite material in terms of the principal material directions (1, 2, 3), where 3 would represent the z-axis, are given by:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (2.22)$$

Q_{ij} are defined in reference [2.4] and are given by:

$$\begin{cases} Q_{11} = E_1(1 - \nu_{23}\nu_{32})/\Delta \\ Q_{22} = E_2(1 - \nu_{31}\nu_{13})/\Delta \\ Q_{44} = \lambda G_{23} \\ Q_{55} = \lambda G_{13} \\ Q_{66} = \lambda G_{12} \\ Q_{12} = E_1(\nu_{21} + \nu_{31}\nu_{23})/\Delta \\ \quad = E_2(\nu_{12} + \nu_{32}\nu_{13})/\Delta \end{cases} \quad (2.23)$$

where $\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}$.

Displacements in the z-direction are not considered, therefore $\nu_{23} = \nu_{32} = \nu_{31} = \nu_{13} = 0$ and (2.23) comes

$$\left\{ \begin{array}{l} Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \\ Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\ Q_{12} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\ Q_{44} = \lambda G_{23} \\ Q_{55} = \lambda G_{13} \\ Q_{66} = G_{12} \end{array} \right. \quad (2.24)$$

Here, E_1 and E_2 are the major and minor Young's moduli; ν_{12} and ν_{21} are the Poisson's ratios; G_{12} is the shear modulus. [2.4] 1 and 2 denote the principal directions of the plate layer. A shear correction factor, $\lambda=5/6$, that accounts for the fact that the shear stresses are not constant across the section, was introduced in (2.24).

Multiplying equation (2.20) by $[T]$ and substituting in equation (2.22), we have

$$[T] \left\{ \begin{array}{l} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{array} \right\} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix} [T] \left\{ \begin{array}{l} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{array} \right\}$$

which is equivalent to

$$\left\{ \begin{array}{l} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{array} \right\} = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix} [T] \left\{ \begin{array}{l} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{array} \right\}$$

by multiplying both members by $[T]^{-1}$. This is equivalent to

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zx} \\ \bar{\sigma}_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix}, \quad (2.25)$$

Considering $\varepsilon_{yz} = \frac{1}{2}\gamma_{yz}$, $\varepsilon_{xz} = \frac{1}{2}\gamma_{xz}$, and $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$, equation (2.25) comes

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{yz} \\ \bar{\sigma}_{zx} \\ \bar{\sigma}_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad (2.26)$$

where $[\bar{Q}] = [T]^{-1} [Q] [T]$ is given by

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \\ \bar{Q}_{16} &= -mn^3Q_{22} + m^3nQ_{11} - mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\ \bar{Q}_{26} &= -m^3nQ_{22} + mn^3Q_{11} + mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\ \bar{Q}_{45} &= (Q_{55} - Q_{44})mn \\ \bar{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\ \bar{Q}_{66} &= Q_{11} + Q_{22} - 2Q_{12}m^2n^2 + Q_{66}(m^2 - n^2)^2 \end{aligned} \quad (2.27)$$

Equation (2.26) can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (2.28)$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

At this point the quantities Q_{ij} and \bar{Q}_{ij} can be determined relating stresses and strains in either coordinate system. Equations (2.28) are the constitutive equations of laminae.

In the following, N is the number of laminae that composes the laminate. For the k^{th} laminae, equation (2.26) can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}_k = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k \quad (2.29)$$

where all matrices must have the subscript k due to the orientation of the particular lamina with respect to the plate x - y coordinates and its unique $[\bar{Q}]$.

Consider equations (2.1) to (2.13), and a laminated plate, elastic, with uniform thickness h , and each layer of the plate is homogeneous and orthotropic; h_k is the vectorial distance from the mid-plane, to the upper surface of the k^{th} lamina.

2.2 - Moment-Curvature and stress relations

The in-plane stress resultants $\{T_x, T_y, T_{xy}\}$ and the moments $\{M_x, M_y, M_{xy}\}$, per unit length are defined by:

$$\{T_x, T_y, T_{xy}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_x, \sigma_y, \tau_{xy}\} dz, \quad (2.30)$$

$$\{M_x, M_y, M_{xy}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_x, \sigma_y, \tau_{xy}\} z dz \quad (2.31)$$

The shear stress resultants are

$$\{Q_x, Q_y\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\tau_{xz}, \tau_{yz}\} dz \quad (2.32)$$

For a laminated plate, the stress components can be integrated across each lamina and added together as follows, employing equations (2.29), (2.5) and (2.6),

$$\begin{aligned} \{T_x, T_y, T_{xy}\} &= \sum_{k=1}^N \int_{h_{k-1}}^{h_k} \{\sigma_x, \sigma_y, \tau_{xy}\} dz = \\ &= \sum_{k=1}^N \left\{ \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \{\varepsilon_x^0, \varepsilon_y^0, \tau_{xy}^0\}_k dz + \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \{k_x, k_y, k_{xy}\}_k z dz \right\} \end{aligned} \quad (2.33)$$

Since the derivatives of u_0 and v_0 (mid-surface displacements) and the \bar{Q} 's are not functions of z , (2.33) can be rewritten as:

$$\begin{aligned} \{T_x, T_y, T_{xy}\} &= \\ &= \sum_{k=1}^N \left\{ [\bar{Q}]_k \{\varepsilon_x^0, \varepsilon_y^0, \tau_{xy}^0\}_k \int_{h_{k-1}}^{h_k} dz + [\bar{Q}]_k \{k_x, k_y, k_{xy}\}_k \int_{h_{k-1}}^{h_k} z dz \right\} \end{aligned} \quad (2.34)$$

or, in reduced form,

$$[T] = [A][\varepsilon_0] + [B][k] \quad (2.35)$$

where

$$A_{ij} = \sum_{k=1}^N (\bar{Q}_{ij})_k [h_k - h_{k-1}], [i,j = 1,2,6] \quad (2.36)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N (\bar{Q}_{ij})_k [h_k^2 - h_{k-1}^2], [i,j = 1,2,6] \quad (2.37)$$

From equation (2.31),

$$\begin{aligned}
\{M_x, M_y, M_{xy}\} &= \sum_{k=1}^N \int_{h_{k-1}}^{h_k} \{\sigma_x, \sigma_y, \tau_{xy}\} z dz = \\
&= \sum_{k=1}^N \left\{ \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \{\varepsilon_x^0, \varepsilon_y^0, \tau_{xy}^0\}_k z dz + \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \{\kappa_x, \kappa_y, \kappa_{xy}\}_k z^2 dz \right\} \\
&= \sum_{k=1}^N \left\{ [\bar{Q}]_k \{\varepsilon_x^0, \varepsilon_y^0, \tau_{xy}^0\}_k \int_{h_{k-1}}^{h_k} z dz + [\bar{Q}]_k \{\kappa_x, \kappa_y, \kappa_{xy}\}_k \int_{h_{k-1}}^{h_k} z^2 dz \right\} \quad (2.38)
\end{aligned}$$

or, in reduced form¹,

$$[M] = [B][\varepsilon_0] + [D][\kappa] \quad (2.39)$$

Where

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N (\bar{Q}_{ij})_k [h_k^3 - h_{k-1}^3], [i,j = 1,2,6] \quad (2.40)$$

From (2.22), (2.5), and (2.30) to (2.32),

$$\tau_{xz_k} = 2\bar{Q}_{55_k} \varepsilon_{xz} + 2\bar{Q}_{45_k} \varepsilon_{yz}$$

$$\tau_{yz_k} = 2\bar{Q}_{45_k} \varepsilon_{xz} + 2\bar{Q}_{44_k} \varepsilon_{yz}$$

hence

$$Q_x = 2(A_{55} \varepsilon_{xz} + A_{45} \varepsilon_{yz}) \quad (2.41)$$

$$Q_y = 2(A_{45} \varepsilon_{xz} + A_{44} \varepsilon_{yz}) \quad (2.42)$$

Combining (2.32) and (2.36),

$$\begin{cases} [T] = [A][\varepsilon_0] + [B][k] \\ [M] = [B][\varepsilon_0] + [D][k] \end{cases} \quad (2.43)$$

we have

$$\begin{Bmatrix} T \\ M \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \{\varepsilon\} = [E] \{\varepsilon\} \quad (2.44)$$

which is equivalent to

¹ The notation here used for the reduced form is equal to the representation of the mass matrix defined in section 2.3.

$$\begin{bmatrix} T_x \\ T_y \\ T_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \tau_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (2.45)$$

where $[A]$ is the extensional stiffness matrix relating in-plane stress resultants (N 's) to the mid-surface strains (ε_0 's); $[D]$ is the flexural stiffness matrix relating the stress couples (M 's) to the curvatures (k 's), and $[B]$ is the bending-stretching matrix and relates M 's to ε_0 's and N 's to k 's. The model presented is valid for laminated plates which may be not symmetrical about their mid-surface plane.

2.3 - Equations of Motion

The equations of motion of the plate are derived by equating the sum of the virtual work of the inertia forces, of the elastic restoring forces, and of external forces to zero. In-plane and transverse external forces are considered.

Combining equations (2. 4), (2. 5) and (2. 6), the strains are expressed as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -z & 0 & 0 \\ 0 & 1 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & -z \end{bmatrix} \{\varepsilon\} \quad (2.46)$$

Where

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \quad (2.47)$$

The linear membrane and bending strains, $\{\varepsilon_0^p\}$ and $\{\varepsilon_0^b\}$, and the geometrically non-linear membrane strain, $\{\varepsilon_L^p\}$, are defined as

$$\{\boldsymbol{\varepsilon}_0^p\} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix}, \{\boldsymbol{\varepsilon}_0^b\} = \begin{Bmatrix} -\frac{\partial\Phi_y^0}{\partial x} \\ \frac{\partial\Phi_x^0}{\partial y} \\ -\frac{\partial\Phi_y^0}{\partial y} + \frac{\partial\Phi_x^0}{\partial x} \end{Bmatrix}, \{\boldsymbol{\varepsilon}_L^p\} = \begin{Bmatrix} (w_{,x})^2/2 \\ (w_{,x})^2/2 \\ w_{,x}w_{,y} \end{Bmatrix} \quad (2.48)$$

Or, in terms of shape functions and generalized displacements, using relation (2.8),

$$\{\boldsymbol{\varepsilon}_0^p\} = \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \quad (2.49)$$

$$\{\boldsymbol{\varepsilon}_0^b\} = \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \quad (2.50)$$

$$\{\boldsymbol{\varepsilon}_L^p\} = \begin{bmatrix} \frac{1}{2}\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} \\ \frac{1}{2}\{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \end{bmatrix} \quad (2.51)$$

The transverse shear strains, $\{\gamma_{zx}\}$ and $\{\gamma_{yz}\}$ are defined as

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} w_{,y}^0 - \Phi_x \\ w_{,x}^0 + \Phi_y \end{Bmatrix} = \begin{bmatrix} \{N_{,y}^w\}^T & 0 & -\{N^{\Phi_x}\}^T \\ \{N_{,x}^w\}^T & \{N^{\Phi_y}\}^T & 0 \end{bmatrix} \begin{Bmatrix} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \quad (2.52)$$

The principle of virtual work states that:

$$\delta W_{in} + \delta W_v + \delta W_{ex} = 0 \quad (2.53)$$

where δW_{in} , δW_v and δW_{ex} are, respectively, the work done by the inertia, internal and external forces due to virtual displacements $\{\delta d\}$. $\{\delta d\}$ is given by

$$\{\delta d\} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \\ \delta \Phi_x \\ \delta \Phi_y \end{Bmatrix} = [N] \{\delta q\} \quad (2.54)$$

Making use of d'Alembert principle, we obtain the following expression for the virtual work of the inertia forces:

$$\delta W_{in} = -\rho h \int_{\Omega} \{\delta d\}^T \{\ddot{d}\} d\Omega = \{\delta q\}^T \left(-\rho h \int_{\Omega} [N]^T [N] d\Omega \{\ddot{q}\} \right) = -\{\delta q\}^T [M] \{\ddot{q}\} \quad (2.55)$$

Where $[M]$ is the mass matrix:

$$[M] = \rho h \int_{\Omega} [N]^T [N] d\Omega = \begin{bmatrix} \rho h \int_{\Omega} \{N^u\}^T \{N^u\}^T d\Omega & 0 & 0 & 0 & 0 \\ 0 & \rho h \int_{\Omega} \{N^v\}^T \{N^v\}^T d\Omega & 0 & 0 & 0 \\ 0 & 0 & \rho h \int_{\Omega} \{N^w\}^T \{N^w\}^T d\Omega & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho h^3}{12} \int_{\Omega} \{N^{\Phi_x}\}^T \{N^{\Phi_x}\}^T d\Omega & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho h^3}{12} \int_{\Omega} \{N^{\Phi_y}\}^T \{N^{\Phi_y}\}^T d\Omega \end{bmatrix} \quad (2.56)$$

where ρ is the mass density of the material that constitutes the plate, $\{\ddot{q}\} = \frac{d^2 \{q\}}{dt^2}$

and Ω represents the area of the plate.

This is equivalent to, in simplified notation,

$$[M] = \begin{bmatrix} [M_p] & 0 & 0 & 0 & 0 \\ 0 & [M_p] & 0 & 0 & 0 \\ 0 & 0 & [M_b] & 0 & 0 \\ 0 & 0 & 0 & [M_{Ry}] & 0 \\ 0 & 0 & 0 & 0 & [M_{Rx}] \end{bmatrix} \quad (2.57)$$

where $[M_p]$ and $[M_b]$ are the in-plane and out-of-plane inertia matrices, $[M_{Ry}]$ and $[M_{Rx}]$ are due to the rotatory inertia.

The variation δW_v may be expressed as:

$$\delta W_v = -\delta \int_{\Omega} \{\varepsilon\}^T \begin{Bmatrix} T \\ M \end{Bmatrix} d\Omega - \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega \quad (2.58)$$

Substituting equations (2.47) and (2.44) in (2.58),

$$\delta W_v = -\delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T \right) [E] \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega - \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega \quad (2.59)$$

$$= -\delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \right) \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega - \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega$$

$$= -\delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega$$

$$- \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega =$$

$$= \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \\ \delta \phi_x \\ \delta \phi_y \end{Bmatrix}^T \int_{\Omega} [N]^T \begin{Bmatrix} P_u(x, y, t) \\ P_v(x, y, t) \\ P_w(x, y, t) \\ 0 \\ 0 \end{Bmatrix} d\Omega \quad (2.60)$$

Where $P_u(x, y, t)$, $P_v(x, y, t)$ and $P_w(x, y, t)$ are the distributed forces (N/m^2) applied to the plate in the x , y , z directions, respectively. The linear stiffness matrix $[K_1]$, non-

linear stiffness matrices, $[K_2]$, $[K_3]$ and $[K_4]$ and the vector of external forces $\{P\}$ are defined as follows:

$$\begin{aligned} & \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega - \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega \\ &= \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega - \int_{\Omega} \{\delta\gamma\}^T \{Q\} d\Omega \\ &= \{\delta q\}^T [K_1] \{q\}, \end{aligned} \quad (2.61)$$

$$\begin{aligned} \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega &= \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega \\ &= \{\delta q\}^T [K_2] \{q\}, \end{aligned} \quad (2.62)$$

$$\begin{aligned} \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega &= \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega \\ &= \{\delta q\}^T [K_3] \{q\} \end{aligned} \quad (2.63)$$

$$\begin{aligned} \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega &= \delta \int_{\Omega} \left(\begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega \\ &= \{\delta q\}^T [K_4] \{q\} \end{aligned} \quad (2.64)$$

$$\{\delta q\}^T \int_{\Omega} [N]^T \begin{Bmatrix} P_u(x, y, t) \\ P_v(x, y, t) \\ P_w(x, y, t) \\ 0 \\ 0 \end{Bmatrix} d\Omega = \{\delta q\}^T \{P\}, \quad (2.65)$$

$$\text{where } \{P\} = \begin{Bmatrix} P_u(x, y, t) \\ P_v(x, y, t) \\ P_w(x, y, t) \\ 0 \\ 0 \end{Bmatrix}.$$

Therefore, equation (2. 59) may be expressed as

$$\delta W_v = \{\delta q\}^T \left([K_1] + [K_2] + [K_3] + [K_4] \right) \{q\} \quad (2. 66)$$

The generalized excitation forces δW_{ex} can be expressed in terms of the actual forces and the shape functions of the HFEM, by means of the virtual work executed by these forces. For example, if $P_j(t)$ represents a transverse concentrated force acting at the point $x = x_j$ and $P_d(x, y, t)$ represents a transverse distributed force, the virtual work of the external forces is given by:

$$\delta W_{ex} = \int_{\Omega} \left[P_j(t) \delta(x - x_j) \delta(y - y_j) + P_d(x, y, t) \right] \delta w(x, t) d\Omega = \{\delta q_w\}^T \{P_w(t)\} \quad (2. 67)$$

where $\delta(x - x_j)$ represents a spatial Dirac delta functions given by

$$\begin{aligned} \delta(x - x_j) & \quad x \neq x_j \\ \int_0^L \delta(x - x_j) dx & = 1 \end{aligned} \quad (2. 68)$$

So that $P_j(t) \delta(x - x_j) \delta(y - y_j)$ has units of distributed force (N/m^2).

Substituting equations (2. 61)-(2. 65) into equation (2. 60) and allowing the virtual generalized displacements to be arbitrary, gives the time domain equations of motion in generalized coordinates:

$$-\{\delta q\}^T [M] \{\ddot{q}\} - \{\delta q\}^T \left([K_1] + [K_2] + [K_3] + [K_4] \right) \{q\} + \{\delta q\}^T \{P(t)\} = 0 \quad (2. 69)$$

This is equivalent to,

$$\begin{aligned}
& \{\delta q\}^T \left(-[M]\{\ddot{q}\} - ([K_1] + [K_2] + [K_3] + [K_4])\{q\} + \{P(t)\} \right) = 0 \\
& \Leftrightarrow [M]\{\ddot{q}\} + [K_1]\{q\} + ([K_2] + [K_3] + [K_4])\{q\} = \{P(t)\} \\
& \Leftrightarrow [M]\{\ddot{q}\} + [Kl]\{q\} + [Knl]\{q\} = \{P\} \tag{2.70}
\end{aligned}$$

In a more detailed form, equation (2.70) may be written as follows:

$$\begin{aligned}
& \begin{bmatrix} [M_p] & 0 & 0 & 0 & 0 \\ 0 & [M_p] & 0 & 0 & 0 \\ 0 & 0 & [M_b] & 0 & 0 \\ 0 & 0 & 0 & [M_{Ry}] & 0 \\ 0 & 0 & 0 & 0 & [M_{Rx}] \end{bmatrix} \begin{Bmatrix} \ddot{q}_u \\ \ddot{q}_v \\ \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} + \\
& + \begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] & 0 & [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{p21}] & [K_1^{p22}] & 0 & [K_1^{pb21}] & [K_1^{pb22}] \\ 0 & 0 & [K_1^{\gamma11}] & [K_1^{\gamma12}] & [K_1^{\gamma13}] \\ [K_1^{bp11}] & [K_1^{bp12}] & [K_1^{\gamma21}] & [K_1^{b11}] + [K_1^{\gamma22}] & [K_1^{b12}] + [K_1^{\gamma23}] \\ [K_1^{bp21}] & [K_1^{bp22}] & [K_1^{\gamma31}] & [K_1^{b21}] + [K_1^{\gamma32}] & [K_1^{b22}] + [K_1^{\gamma33}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \\
& + \begin{bmatrix} 0 & 0 & [K_2^{11}] & 0 & 0 \\ 0 & 0 & [K_2^{21}] & 0 & 0 \\ [K_3^{11}] & [K_3^{12}] & [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_u\} \\ \{P_v\} \\ \{P_w\} \\ 0 \\ 0 \end{Bmatrix}, \tag{2.71}
\end{aligned}$$

Where

$$\begin{aligned}
& \left[\begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{bmatrix}, \begin{bmatrix} [K_1^{b11}] & [K_1^{b12}] \\ [K_1^{b21}] & [K_1^{b22}] \end{bmatrix}, \begin{bmatrix} [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{pb21}] & [K_1^{pb22}] \end{bmatrix}, \begin{bmatrix} [K_1^{bp11}] & [K_1^{bp12}] \\ [K_1^{bp21}] & [K_1^{bp22}] \end{bmatrix} \right],
\end{aligned}$$

$$\begin{bmatrix} [K_1^{\gamma 11}] & [K_1^{\gamma 12}] & [K_1^{\gamma 13}] \\ [K_1^{\gamma 21}] & [K_1^{\gamma 22}] & [K_1^{\gamma 23}] \\ [K_1^{\gamma 31}] & [K_1^{\gamma 32}] & [K_1^{\gamma 33}] \end{bmatrix}, \begin{bmatrix} [K_2^{11}] \\ [K_2^{21}] \\ [K_2^{31}] \end{bmatrix} \text{ and } \begin{bmatrix} [K_3^{11}] & [K_3^{12}] & [K_3^{13}] \end{bmatrix} \text{ indicate the area}$$

which $[K_1^p], [K_1^b], [K_1^{pb}], [K_1^{bp}], [K_1^\gamma], [K_2]$ and $[K_3]$ occupy, respectively.

$[K_1^p]$ is the in-plane linear stiffness matrix, $[K_1^b]$ is the out-plane linear stiffness matrix, $[K_1^{pb}]$ and $[K_1^{bp}]$ are the in-plane/bending coupling matrices, and $[K_1^\gamma]$ is

the shear linear stiffness matrix; they form the linear $[K_1]$ matrix. $[K_2], [K_3]$ and

$[K_4]$ represent the nonzero part of the non-linear stiffness matrices and

$\{P\} = \{\{P_u\} \quad \{P_v\} \quad \{P_w\} \quad 0 \quad 0\}^T$ is the vector of generalized external forces. These

matrices are defined in the following sections, with the exception of $[K_3]$. As is

demonstrated in reference [2.1], by comparing the form of $[K_3]$ and $[K_2]$,

$$[K_3] = 2[K_2]^T.$$

From equation (2. 71), the system can be split in two parts:

$$\begin{bmatrix} [M_p] & 0 & 0 & 0 & 0 \\ 0 & [M_p] & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_u \\ \ddot{q}_v \\ \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} + \begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] & 0 & [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{p21}] & [K_1^{p22}] & 0 & [K_1^{pb21}] & [K_1^{pb22}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & [K_2^{11}] & 0 & 0 \\ 0 & 0 & [K_2^{21}] & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_u\} \\ \{P_v\} \end{Bmatrix} \quad (2.72)$$

And

$$\begin{bmatrix} [M_b] & 0 & 0 \\ 0 & [M_{Ry}] & 0 \\ 0 & 0 & [M_{Rx}] \end{bmatrix} \begin{Bmatrix} \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & [K_1^{\gamma 11}] & [K_1^{\gamma 12}] & [K_1^{\gamma 13}] \\ [K_1^{bp11}] & [K_1^{bp12}] & [K_1^{\gamma 21}] & [K_1^{b11}] + [K_1^{\gamma 22}] & [K_1^{b12}] + [K_1^{\gamma 23}] \\ [K_1^{bp21}] & [K_1^{bp22}] & [K_1^{\gamma 31}] & [K_1^{b21}] + [K_1^{\gamma 32}] & [K_1^{b22}] + [K_1^{\gamma 33}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} + \begin{bmatrix} [K_3^{11}] & [K_3^{12}] & [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_w\} \\ 0 \\ 0 \end{Bmatrix} \quad (2.73)$$

In not too thick plates and if the in-plane displacements are much smaller than the transverse displacements, the in-plane inertia can be neglected. Therefore, from (2.72), comes

$$\begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] & 0 & [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{p21}] & [K_1^{p22}] & 0 & [K_1^{pb21}] & [K_1^{pb22}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & [K_2^{11}] & 0 & 0 \\ 0 & 0 & [K_2^{21}] & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_u\} \\ \{P_v\} \end{Bmatrix} \quad (2.74)$$

Solving (2.74) for $\begin{Bmatrix} q_u \\ q_v \end{Bmatrix}$, comes

$$\begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{bmatrix}^{-1} \begin{Bmatrix} \{P_u\} \\ \{P_v\} \end{Bmatrix} - \begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{bmatrix}^{-1} \begin{bmatrix} [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{pb21}] & [K_1^{pb22}] \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \\ - \begin{bmatrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{bmatrix}^{-1} \begin{bmatrix} [K_2^{11}] \\ [K_2^{21}] \end{bmatrix} \{q_w\} \quad (2.75)$$

From (2.73),

$$\begin{bmatrix} [M_b] & 0 & 0 \\ 0 & [M_{Ry}] & 0 \\ 0 & 0 & [M_{Rx}] \end{bmatrix} \begin{Bmatrix} \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ [K_1^{bp11}] & [K_1^{bp12}] & 0 & 0 & 0 \\ [K_1^{bp21}] & [K_1^{bp22}] & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & [K_1^{\gamma11}] & [K_1^{\gamma12}] & [K_1^{\gamma13}] \\ 0 & 0 & [K_1^{\gamma21}] & [K_1^{b11}] + [K_1^{\gamma22}] & [K_1^{b12}] + [K_1^{\gamma23}] \\ 0 & 0 & [K_1^{\gamma31}] & [K_1^{b21}] + [K_1^{\gamma32}] & [K_1^{b22}] + [K_1^{\gamma33}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} +$$

$$+ \begin{bmatrix} [K_3^{11}] & [K_3^{12}] & [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_w\} \\ 0 \\ 0 \end{Bmatrix} \Leftrightarrow$$

$$\begin{aligned}
& \begin{bmatrix} [M_b] & 0 & 0 \\ 0 & [M_{Ry}] & 0 \\ 0 & 0 & [M_{Rx}] \end{bmatrix} \begin{Bmatrix} \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} + \begin{bmatrix} 0 \\ [K_1^{bp11}] & [K_1^{bp12}] \\ [K_1^{bp21}] & [K_1^{bp22}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} + \\
& + \begin{bmatrix} 0 & 0 & [K_1^{\gamma11}] & [K_1^{\gamma12}] & [K_1^{\gamma13}] \\ 0 & 0 & [K_1^{\gamma21}] & [K_1^{b11}] + [K_1^{\gamma22}] & [K_1^{b12}] + [K_1^{\gamma23}] \\ 0 & 0 & [K_1^{\gamma31}] & [K_1^{b21}] + [K_1^{\gamma32}] & [K_1^{b22}] + [K_1^{\gamma33}] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} + \\
& + \begin{bmatrix} [K_3^{11}] & [K_3^{12}] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} + \\
& + \begin{bmatrix} 0 & 0 & [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \\ q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_w\} \\ 0 \\ 0 \end{Bmatrix} \Leftrightarrow
\end{aligned}$$

$$\begin{bmatrix} [M_b] & 0 & 0 \\ 0 & [M_{Ry}] & 0 \\ 0 & 0 & [M_{Rx}] \end{bmatrix} \begin{Bmatrix} \ddot{q}_w \\ \ddot{q}_{\Phi_y} \\ \ddot{q}_{\Phi_x} \end{Bmatrix} +$$

$$\begin{bmatrix} [K_1^{\gamma11}] & [K_1^{\gamma12}] & [K_1^{\gamma13}] \\ [K_1^{\gamma21}] & [K_1^{b11}] + [K_1^{\gamma22}] & [K_1^{b12}] + [K_1^{\gamma23}] \\ [K_1^{\gamma31}] & [K_1^{b21}] + [K_1^{\gamma32}] & [K_1^{b22}] + [K_1^{\gamma33}] \end{bmatrix} \begin{Bmatrix} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} +$$

$$\begin{bmatrix} [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \begin{Bmatrix} \{P_w\} \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} [[K_3^{11}] & [K_3^{12}]] \\ [[K_1^{bp11}] & [K_1^{bp12}]] \\ [[K_1^{bp21}] & [K_1^{bp22}]] \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix}$$

(2. 76)

Replacing (2.75) in (2. 76) the reduced equations of motion are obtained.

From (2. 70), the mass matrix $[M]$, the linear stiffness matrix $[Kl]$ and the non-linear stiffness matrix $[Knl]$ are given by:

$$[M] = \begin{bmatrix} [M_b] & 0 & 0 \\ 0 & [M_{Ry}] & 0 \\ 0 & 0 & [M_{Rx}] \end{bmatrix},$$

$$[Kl] = \begin{bmatrix} [K_1^{\gamma 11}] & [K_1^{\gamma 12}] & [K_1^{\gamma 13}] \\ [K_1^{\gamma 21}] & [K_1^{b11}] + [K_1^{\gamma 22}] & [K_1^{b12}] + [K_1^{\gamma 23}] \\ [K_1^{\gamma 31}] & [K_1^{b21}] + [K_1^{\gamma 32}] & [K_1^{b22}] + [K_1^{\gamma 33}] \end{bmatrix},$$

$$[Knl] = \begin{bmatrix} 0 & 0 & [K_2^{31}] + [K_3^{13}] + [K_4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the latter with terms which are either linear or quadratic functions of $\{q_w\}$. The

vector of external forces $\{P\}$ is given by

$$\{P\} = \begin{Bmatrix} \{P_w\} - [K_3^{11}]\{q_u\} - [K_3^{12}]\{q_v\} \\ -[K_1^{bp11}]\{q_u\} - [K_1^{bp12}]\{q_v\} \\ -[K_1^{bp21}]\{q_u\} - [K_1^{bp22}]\{q_v\} \end{Bmatrix} \quad (2. 77)$$

If a distributed force that impinges on the plate's surface in the z direction is considered, then $\{P_u\} = \{P_v\} = 0$. Therefore, from equation (2.72), the new force vector is given by

$$\{P\} = \begin{pmatrix} \left\{ P_w \right\} + \left[\begin{matrix} [K_3^{11}] & [K_3^{12}] \\ [K_3^{21}] & [K_3^{22}] \end{matrix} \right] \left[\begin{matrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{matrix} \right]^{-1} \left(\left[\begin{matrix} [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{pb21}] & [K_1^{pb22}] \end{matrix} \right] \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} + \left[\begin{matrix} [K_2^{11}] \\ [K_2^{21}] \end{matrix} \right] \begin{Bmatrix} q_w \end{Bmatrix} \right) \\ + \left[\begin{matrix} [K_1^{bp11}] & [K_1^{bp12}] \\ [K_1^{bp21}] & [K_1^{bp22}] \end{matrix} \right] \left[\begin{matrix} [K_1^{p11}] & [K_1^{p12}] \\ [K_1^{p21}] & [K_1^{p22}] \end{matrix} \right]^{-1} \left(\left[\begin{matrix} [K_1^{pb11}] & [K_1^{pb12}] \\ [K_1^{pb21}] & [K_1^{pb22}] \end{matrix} \right] \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} - \left[\begin{matrix} [K_2^{11}] \\ [K_2^{21}] \end{matrix} \right] \begin{Bmatrix} q_w \end{Bmatrix} \right) \end{pmatrix} \quad (2.78)$$

2.4 - Derivation of the Matrices Used in the Equations of Motion

2.4.1 - Linear stiffness matrix $[K_1]$

From equation (2. 61),

$$\begin{aligned} \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega &= \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega = \\ &= \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [A] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [B] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [B] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [D] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} \right) d\Omega \end{aligned} \quad (2.79)$$

The linear $[K_1]$ matrix consists of four independent parts. The in plane stiffness matrix, $[K_1^p]$, the bending stiffness matrix, $[K_1^b]$, and the coupling in-plane/bending matrices $[K_1^{pb}]$, $[K_1^{bp}]$ which are obtained from (2. 79).

Therefore,

$$\int_{\Omega} \delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [A] \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} d\Omega = \int_{\Omega} \left(\delta \begin{bmatrix} \begin{Bmatrix} N_{,x}^u \\ 0 \\ N_{,y}^u \\ N_{,x}^u \end{Bmatrix}^T & 0 \\ 0 & \begin{Bmatrix} N_{,y}^u \\ N_{,x}^u \end{Bmatrix}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \right)^T [A] \begin{bmatrix} \begin{Bmatrix} N_{,x}^u \\ 0 \\ N_{,y}^u \\ N_{,x}^u \end{Bmatrix}^T & 0 \\ 0 & \begin{Bmatrix} N_{,y}^u \\ N_{,x}^u \end{Bmatrix}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \right) d\Omega =$$

$$\begin{aligned}
&= \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
&= \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} \{N_{,x}^u\} & 0 & \{N_{,y}^u\} \\ 0 & \{N_{,y}^u\} & \{N_{,x}^u\} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
&= \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} A_{11} \{N_{,x}^u\} \{N_{,x}^u\}^T + A_{16} \left(\{N_{,y}^u\} \{N_{,x}^u\}^T + \{N_{,x}^u\} \{N_{,y}^u\}^T \right) + A_{66} \{N_{,y}^u\} \{N_{,y}^u\}^T \\ A_{12} \{N_{,y}^u\} \{N_{,x}^u\}^T + A_{16} \{N_{,x}^u\} \{N_{,x}^u\}^T + A_{26} \{N_{,y}^u\} \{N_{,y}^u\}^T + A_{66} \{N_{,x}^u\} \{N_{,y}^u\}^T \\ A_{12} \{N_{,x}^u\} \{N_{,y}^u\}^T + A_{26} \{N_{,y}^u\} \{N_{,y}^u\}^T + A_{16} \{N_{,x}^u\} \{N_{,x}^u\}^T + A_{66} \{N_{,y}^u\} \{N_{,x}^u\}^T \\ A_{22} \{N_{,y}^u\} \{N_{,y}^u\}^T + A_{26} \left(\{N_{,x}^u\} \{N_{,y}^u\}^T + \{N_{,y}^u\} \{N_{,x}^u\}^T \right) + A_{66} \{N_{,x}^u\} \{N_{,x}^u\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
&= \{\delta q\}^T [K_1^p] \{q\} \tag{2.80}
\end{aligned}$$

The bending stiffness matrix $[K_1^b]$ is derived considering the virtual

$$\text{work } \int_{\Omega} \delta \{\varepsilon_0^b\}^T [D] \{\varepsilon_0^b\} d\Omega. \text{ Replacing } \{\varepsilon_0^b\} \text{ by } \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \text{ in the}$$

integral, comes

$$\begin{aligned}
& \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\} & 0 & -\{N_{,y}^{\Phi_y}\} \\ 0 & \{N_{,y}^{\Phi_x}\} & \{N_{,x}^{\Phi_x}\} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} D_{11} \{N_{,x}^{\Phi_y}\} \{N_{,x}^{\Phi_y}\}^T + D_{16} \left(\{N_{,y}^{\Phi_y}\} \{N_{,x}^{\Phi_y}\}^T + \{N_{,x}^{\Phi_y}\} \{N_{,y}^{\Phi_y}\}^T \right) + D_{66} \{N_{,y}^{\Phi_y}\} \{N_{,y}^{\Phi_y}\}^T \\ -D_{12} \{N_{,y}^{\Phi_x}\} \{N_{,x}^{\Phi_y}\}^T - D_{16} \{N_{,x}^{\Phi_x}\} \{N_{,y}^{\Phi_y}\}^T - D_{26} \{N_{,y}^{\Phi_x}\} \{N_{,y}^{\Phi_y}\}^T - D_{66} \{N_{,x}^{\Phi_x}\} \{N_{,y}^{\Phi_y}\}^T \\ -D_{12} \{N_{,x}^{\Phi_y}\} \{N_{,y}^{\Phi_x}\}^T - D_{26} \{N_{,y}^{\Phi_x}\} \{N_{,y}^{\Phi_x}\}^T - D_{16} \{N_{,x}^{\Phi_y}\} \{N_{,x}^{\Phi_x}\}^T - D_{66} \{N_{,y}^{\Phi_x}\} \{N_{,x}^{\Phi_x}\}^T \\ D_{22} \{N_{,y}^{\Phi_x}\} \{N_{,y}^{\Phi_x}\}^T + D_{26} \left(\{N_{,x}^{\Phi_x}\} \{N_{,y}^{\Phi_x}\}^T + \{N_{,y}^{\Phi_x}\} \{N_{,x}^{\Phi_x}\}^T \right) + D_{66} \{N_{,x}^{\Phi_x}\} \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \left[K_1^b \right] \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix}
\end{aligned} \tag{2.81}$$

Matrix $[K_1^{pb}]$ is obtained from the integral $\int_{\Omega} \delta \{\varepsilon_0^p\}^T [B] \{\varepsilon_0^b\} d\Omega$. Replacing $\{\varepsilon_0^p\}^T$ and $\{\varepsilon_0^b\}$ in the integral, comes

$$\begin{aligned}
& \int_{\Omega} \left(\delta \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \right)^T \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} d\Omega = \\
& = \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} \{N_{,x}^u\} & 0 & \{N_{,y}^u\} \\ 0 & \{N_{,y}^u\} & \{N_{,x}^u\} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} (\{N_{,x}^u\} B_{11} + \{N_{,y}^u\} B_{16}) (-\{N_{,x}^{\Phi_y}\}^T) + (\{N_{,x}^u\} B_{16} + \{N_{,y}^u\} B_{66}) (-\{N_{,y}^{\Phi_y}\}^T) \\ (\{N_{,y}^u\} B_{12} + \{N_{,x}^u\} B_{16}) (-\{N_{,x}^{\Phi_y}\}^T) + (\{N_{,y}^u\} B_{26} + \{N_{,x}^u\} B_{66}) (-\{N_{,y}^{\Phi_y}\}^T) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& \left(\{N_{,x}^u\} \mathbf{B}_{12} + \{N_{,y}^u\} \mathbf{B}_{26} \right) \{N_{,y}^{\Phi_x}\}^T + \left(\{N_{,x}^u\} \mathbf{B}_{16} + \{N_{,y}^u\} \mathbf{B}_{66} \right) \{N_{,x}^{\Phi_x}\}^T \Big] d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& \left(\{N_{,y}^u\} \mathbf{B}_{22} + \{N_{,x}^u\} \mathbf{B}_{26} \right) \{N_{,y}^{\Phi_x}\}^T + \left(\{N_{,y}^u\} \mathbf{B}_{26} + \{N_{,x}^u\} \mathbf{B}_{66} \right) \{N_{,x}^{\Phi_x}\}^T \Big] d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} -\mathbf{B}_{11} \{N_{,x}^u\} \{N_{,x}^{\Phi_y}\}^T - \mathbf{B}_{16} \left(\{N_{,y}^u\} \{N_{,x}^{\Phi_y}\}^T + \{N_{,x}^u\} \{N_{,y}^{\Phi_y}\}^T \right) - \mathbf{B}_{66} \{N_{,y}^u\} \{N_{,y}^{\Phi_y}\}^T \\ -\mathbf{B}_{12} \{N_{,y}^u\} \{N_{,x}^{\Phi_y}\}^T - \mathbf{B}_{16} \{N_{,y}^u\} \{N_{,x}^{\Phi_y}\}^T - \mathbf{B}_{26} \{N_{,y}^u\} \{N_{,y}^{\Phi_y}\}^T - \mathbf{B}_{66} \{N_{,y}^u\} \{N_{,y}^{\Phi_y}\}^T \\ \mathbf{B}_{12} \{N_{,x}^u\} \{N_{,y}^{\Phi_x}\}^T + \mathbf{B}_{26} \{N_{,y}^u\} \{N_{,y}^{\Phi_x}\}^T + \mathbf{B}_{16} \{N_{,x}^u\} \{N_{,x}^{\Phi_x}\}^T + \mathbf{B}_{66} \{N_{,y}^u\} \{N_{,x}^{\Phi_x}\}^T \\ \mathbf{B}_{22} \{N_{,y}^u\} \{N_{,y}^{\Phi_x}\}^T + \mathbf{B}_{26} \left(\{N_{,x}^u\} \{N_{,y}^{\Phi_x}\}^T + \{N_{,y}^u\} \{N_{,x}^{\Phi_x}\}^T \right) + \mathbf{B}_{66} \{N_{,x}^u\} \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T \left[\mathbf{K}_1^{pb} \right] \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \tag{2.82}
\end{aligned}$$

Matrix $\left[\mathbf{K}_1^{bp} \right]$ is obtained from the integral $\int_{\Omega} \delta \{ \varepsilon_0^b \}^T [B] \{ \varepsilon_0^b \} d\Omega$. Replacing

$\{ \varepsilon_0^b \}^T$ and $\{ \varepsilon_0^b \}$ in the integral, comes

$$\begin{aligned}
& \int_{\Omega} \left(\delta \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \right)^T \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{16} \\ \mathbf{B}_{12} & \mathbf{B}_{22} & \mathbf{B}_{26} \\ \mathbf{B}_{16} & \mathbf{B}_{26} & \mathbf{B}_{66} \end{bmatrix} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} d\Omega = \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\} & 0 & -\{N_{,y}^{\Phi_y}\} \\ 0 & \{N_{,y}^{\Phi_x}\} & \{N_{,x}^{\Phi_x}\} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{16} \\ \mathbf{B}_{12} & \mathbf{B}_{22} & \mathbf{B}_{26} \\ \mathbf{B}_{16} & \mathbf{B}_{26} & \mathbf{B}_{66} \end{bmatrix} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \int_{\Omega} \left[\left(-\{N_{,x}^{\Phi_y}\} \mathbf{B}_{11} - \{N_{,y}^{\Phi_y}\} \mathbf{B}_{16} \right) \{N_{,x}^u\}^T + \left(-\{N_{,x}^{\Phi_y}\} \mathbf{B}_{16} - \{N_{,y}^{\Phi_y}\} \mathbf{B}_{66} \right) \{N_{,y}^u\}^T \right. \\
& \quad \left. + \left(\{N_{,y}^{\Phi_x}\} \mathbf{B}_{12} + \{N_{,x}^{\Phi_x}\} \mathbf{B}_{16} \right) \{N_{,x}^u\}^T + \left(\{N_{,y}^{\Phi_x}\} \mathbf{B}_{26} + \{N_{,x}^{\Phi_x}\} \mathbf{B}_{66} \right) \{N_{,y}^u\}^T \right] d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} =
\end{aligned}$$

$$\begin{aligned}
& \left(-\{N_{,x}^{\Phi_y}\} B_{12} - \{N_{,y}^{\Phi_y}\}^T B_{26} \right) \{N_{,y}^u\}^T + \left(-\{N_{,x}^{\Phi_y}\} B_{16} - \{N_{,y}^{\Phi_y}\} B_{66} \right) \{N_{,x}^u\}^T \Big] d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
& \left(\{N_{,y}^{\Phi_x}\} B_{22} + \{N_{,x}^{\Phi_x}\}^T B_{26} \right) \{N_{,y}^u\}^T + \left(\{N_{,y}^{\Phi_x}\} B_{26} + \{N_{,x}^{\Phi_x}\} B_{66} \right) \{N_{,x}^u\}^T \Big] \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T \int_{\Omega} \begin{bmatrix} -B_{11} \{N_{,x}^{\Phi_y}\} \{N_{,x}^u\}^T - B_{16} \left(\{N_{,y}^{\Phi_y}\} \{N_{,x}^u\}^T + \{N_{,x}^{\Phi_y}\} \{N_{,y}^u\}^T \right) - B_{66} \{N_{,y}^{\Phi_y}\} \{N_{,y}^u\}^T \\ B_{12} \{N_{,y}^{\Phi_x}\} \{N_{,x}^u\}^T + B_{16} \{N_{,x}^{\Phi_x}\} \{N_{,x}^u\}^T + B_{26} \{N_{,y}^{\Phi_x}\} \{N_{,y}^u\}^T + B_{66} \{N_{,x}^{\Phi_x}\} \{N_{,x}^u\}^T \\ -B_{12} \{N_{,x}^{\Phi_y}\} \{N_{,y}^u\}^T - B_{26} \{N_{,y}^{\Phi_y}\} \{N_{,y}^u\}^T - B_{16} \{N_{,x}^{\Phi_y}\} \{N_{,x}^u\}^T - B_{66} \{N_{,y}^{\Phi_y}\} \{N_{,x}^u\}^T \\ B_{22} \{N_{,y}^{\Phi_x}\} \{N_{,y}^u\}^T + B_{26} \left(\{N_{,x}^{\Phi_x}\} \{N_{,y}^u\}^T + \{N_{,y}^{\Phi_x}\} \{N_{,x}^u\}^T \right) + B_{66} \{N_{,x}^{\Phi_x}\} \{N_{,x}^u\}^T \end{bmatrix} d\Omega \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} = \\
& = \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T [K_1^{bp}] \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \tag{2.83}
\end{aligned}$$

One verifies, as expected, that $[K_1^{bp}] = [K_1^{pb}]^T$.

2.4.2 – Non-linear stiffness matrix $[K_2]$ and $[K_3]$

In this section the non-linear stiffness matrix $[K_2]$ is derived. $[K_3]$ is proved to be twice the transpose of $[K_2]$, i.e., $[K_3] = 2[K_2]^T$ [2.1]. Both $[K_2]$ and $[K_3]$ are linear functions of the generalised displacements.

Considering equation (2.62),

$$\begin{aligned}
& \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega = \{\delta q\}^T [K_2] \{q\}. \text{ Therefore,} \\
& \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega = \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega = \\
& = \int_{\Omega} \left(\{\delta \varepsilon_0^p\}^T [A] \{\varepsilon_L^p\} + \{\delta \varepsilon_0^b\}^T [B] \{\varepsilon_L^p\} \right) d\Omega \tag{2.84}
\end{aligned}$$

From the definition of $\{\varepsilon_0^p\}$, $\{\varepsilon_0^b\}$ and $\{\varepsilon_L^p\}$, equations (2.49), (2.50) and (2.51), results that

$$\begin{aligned}
& \int_{\Omega} \left(\delta \begin{bmatrix} \begin{bmatrix} \{N_{,x}^u\}^T & 0 \\ 0 & \{N_{,y}^u\}^T \\ \{N_{,y}^u\}^T & \{N_{,x}^u\}^T \end{bmatrix} \begin{Bmatrix} q_u \\ q_v \end{Bmatrix} \\ + \delta \begin{bmatrix} \begin{bmatrix} -\{N_{,x}^{\Phi_y}\}^T & 0 \\ 0 & \{N_{,y}^{\Phi_x}\}^T \\ -\{N_{,y}^{\Phi_y}\}^T & \{N_{,x}^{\Phi_x}\}^T \end{bmatrix} \begin{Bmatrix} q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} \\ \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \end{bmatrix} + \\
& \left. \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} \\ \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \end{bmatrix} \right) d\Omega \\
& = \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T [K_2^{pl}] \{q_w\} + \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T [K_2^{bpl}] \{q_w\} \quad (2.85)
\end{aligned}$$

Evaluating each term separately,

$$\begin{aligned}
& \begin{Bmatrix} \delta q_u \\ \delta q_v \end{Bmatrix}^T [K_2^{pl}] \{q_w\} = \\
& = \int_{\Omega} \left(\{ \delta q_u \}^T \{N_{,x}^u\} A_{11} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,y}^u\} A_{12} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \right. \\
& + \{ \delta q_u \}^T \{N_{,y}^u\} A_{16} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,x}^u\} A_{16} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \\
& + \{ \delta q_u \}^T \{N_{,x}^u\} A_{12} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,y}^u\} A_{22} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \{ \delta q_u \}^T \{N_{,y}^u\} A_{26} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,x}^u\} A_{26} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \{ \delta q_u \}^T \{N_{,x}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,y}^u\} A_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& \left. + \{ \delta q_u \}^T \{N_{,y}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} + \{ \delta q_v \}^T \{N_{,x}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \right) d\Omega =
\end{aligned}$$

Separating the terms with $\{q_u\}^T$ and $\{q_v\}^T$,

$$\begin{aligned}
& = \{ \delta q_u \}^T \int_{\Omega} \frac{1}{2} \left(\{N_{,x}^u\} A_{11} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,y}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \{N_{,x}^u\} A_{12} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,y}^u\} A_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \{N_{,x}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2 \{N_{,y}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \{q_w\} +
\end{aligned}$$

$$\begin{aligned}
& + \{\delta q_v\}^T \int_{\Omega} \frac{1}{2} \left(\{N_{,y}^u\} A_{12} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,x}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \{N_{,y}^u\} A_{22} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,x}^u\} A_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \{N_{,y}^u\} A_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2 \{N_{,x}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \{q_w\} \\
& = \{\delta q_u\}^T [\bar{U}] \{q_w\} + \{\delta q_v\}^T [\bar{V}] \{q_w\} \tag{2.86}
\end{aligned}$$

Where

$$\begin{aligned}
[\bar{U}] & = \int_{\Omega} \frac{1}{2} \left(\{N_{,x}^u\} A_{11} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,y}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \{N_{,x}^u\} A_{12} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,y}^u\} A_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \{N_{,x}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2 \{N_{,y}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega = \\
& = \int_{\Omega} \frac{1}{2} \left(\left(\{N_{,x}^u\} A_{11} + \{N_{,y}^u\} A_{16} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \left(\{N_{,x}^u\} A_{12} + \{N_{,y}^u\} A_{26} \right) \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \left(\{N_{,x}^u\} A_{16} + \{N_{,y}^u\} A_{66} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \tag{2.87}
\end{aligned}$$

$$\begin{aligned}
[\bar{V}] & = \int_{\Omega} \frac{1}{2} \left(\{N_{,y}^u\} A_{12} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,x}^u\} A_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \{N_{,y}^u\} A_{22} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,x}^u\} A_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \{N_{,y}^u\} A_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2 \{N_{,x}^u\} A_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega = \\
& = \int_{\Omega} \frac{1}{2} \left(\left(\{N_{,y}^u\} A_{12} + \{N_{,x}^u\} A_{16} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \left(\{N_{,y}^u\} A_{22} + \{N_{,x}^u\} A_{26} \right) \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& \left. + 2 \left(\{N_{,y}^u\} A_{26} + \{N_{,x}^u\} A_{66} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \tag{2.88}
\end{aligned}$$

The term $\begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T [K_2^{bpl}] \{q_w\}$, comes

$$\begin{aligned}
& \begin{Bmatrix} \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{Bmatrix}^T [K_2^{bpl}] \{q_w\} = \\
& = \int_{\Omega} \left(- \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,x}^{\Phi_y}\} B_{11} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \right. \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,y}^{\Phi_x}\} B_{12} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} - \\
& - \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,y}^{\Phi_y}\} B_{16} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,x}^{\Phi_x}\} B_{16} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} - \\
& - \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,x}^{\Phi_y}\} B_{12} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,y}^{\Phi_x}\} B_{22} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} - \\
& - \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,y}^{\Phi_y}\} B_{26} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,x}^{\Phi_x}\} B_{26} \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} - \\
& - \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,x}^{\Phi_y}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,y}^{\Phi_x}\} B_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} - \\
& - \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \{N_{,y}^{\Phi_y}\} B_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \{N_{,x}^{\Phi_x}\} B_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \Big) d\Omega
\end{aligned}$$

Separating the terms with $\begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T$ and $\begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T$,

$$\begin{aligned}
& = \begin{Bmatrix} \delta q_{\Phi_y} \end{Bmatrix}^T \int_{\Omega} \frac{1}{2} \left(- \{N_{,x}^{\Phi_y}\} B_{11} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T - \{N_{,y}^{\Phi_y}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T - \right. \\
& - \{N_{,x}^{\Phi_y}\} B_{12} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T - \{N_{,y}^{\Phi_y}\} B_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T - \\
& - 2 \{N_{,x}^{\Phi_y}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T - 2 \{N_{,y}^{\Phi_y}\} B_{66} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \Big) d\Omega \{q_w\} + \\
& + \begin{Bmatrix} \delta q_{\Phi_x} \end{Bmatrix}^T \frac{1}{2} \int_{\Omega} \left(\{N_{,y}^{\Phi_x}\} B_{12} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,x}^{\Phi_x}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\
& + \{N_{,y}^{\Phi_x}\} B_{22} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,x}^{\Phi_x}\} B_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
& + 2 \{N_{,y}^{\Phi_x}\} B_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2 \{N_{,x}^{\Phi_x}\} B_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \Big) d\Omega \{q_w\} =
\end{aligned}$$

$$= \{\delta q_{\Phi_y}\}^T [\bar{Y}] \{q_w\} + \{\delta q_{\Phi_x}\}^T [\bar{Z}] \{q_w\} \quad (2. 89)$$

where

$$\begin{aligned} [\bar{Y}] &= \int_{\Omega} \frac{1}{2} \left(-\{N_{,x}^{\phi_y}\} B_{11} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T - \{N_{,y}^{\phi_y}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T - \right. \\ &\quad - \{N_{,x}^{\phi_y}\} B_{12} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T - \{N_{,y}^{\phi_y}\} B_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T - \\ &\quad \left. - 2\{N_{,x}^{\phi_y}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T - 2\{N_{,y}^{\phi_y}\} B_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega = \\ &= \int_{\Omega} \frac{1}{2} \left(\left(-\{N_{,x}^{\phi_y}\} B_{11} - \{N_{,y}^{\phi_y}\} B_{16} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\ &\quad \left. + \left(-\{N_{,x}^{\phi_y}\} B_{12} - \{N_{,y}^{\phi_y}\} B_{26} \right) \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \right. \\ &\quad \left. 2 \left(-\{N_{,x}^{\phi_y}\} B_{16} - \{N_{,y}^{\phi_y}\} B_{66} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \end{aligned} \quad (2. 90)$$

$$\begin{aligned} [\bar{Z}] &= \int_{\Omega} \frac{1}{2} \left(\{N_{,y}^{\Phi_x}\} B_{12} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \{N_{,x}^{\Phi_x}\} B_{16} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\ &\quad \left. + \{N_{,y}^{\Phi_x}\} B_{22} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \{N_{,x}^{\Phi_x}\} B_{26} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \right. \\ &\quad \left. + 2\{N_{,y}^{\Phi_x}\} B_{26} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T + 2\{N_{,x}^{\Phi_x}\} B_{66} \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega = \\ &= \int_{\Omega} \frac{1}{2} \left(\left(\{N_{,y}^{\Phi_x}\} B_{12} + \{N_{,x}^{\Phi_x}\} B_{16} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \right. \\ &\quad \left. + \left(\{N_{,y}^{\Phi_x}\} B_{22} + \{N_{,x}^{\Phi_x}\} B_{26} \right) \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \right. \\ &\quad \left. + 2 \left(\{N_{,y}^{\Phi_x}\} B_{26} + \{N_{,x}^{\Phi_x}\} B_{66} \right) \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \right) d\Omega \end{aligned} \quad (2. 91)$$

Summing the two integrals obtained, from equation (2. 84) results:

$$\begin{aligned} \{\delta q\}^T [K_2] \{q\} &= \\ &= \{\delta q_u\}^T [\bar{U}] \{q_w\} + \{\delta q_v\}^T [\bar{V}] \{q_w\} + \{\delta q_{\Phi_y}\}^T [\bar{Y}] \{q_w\} + \{\delta q_{\Phi_x}\}^T [\bar{Z}] \{q_w\} = \end{aligned}$$

$$= \begin{Bmatrix} \delta q_u \\ \delta q_v \\ \delta q_{\phi_y} \\ \delta q_{\phi_x} \end{Bmatrix}^T \begin{Bmatrix} [\bar{U}] \\ [\bar{V}] \\ [\bar{Y}] \\ [\bar{Z}] \end{Bmatrix} \{q_w\} = \begin{Bmatrix} \delta q_u \\ \delta q_v \\ \delta q_{\phi_y} \\ \delta q_{\phi_x} \end{Bmatrix}^T [K_2] \{q_w\}$$

The non-zero part of the non-linear stiffness matrix $[K_2]$ is:

$$[K_2] = \begin{Bmatrix} [\bar{U}] \\ [\bar{V}] \\ [\bar{Y}] \\ [\bar{Z}] \end{Bmatrix} \quad (2.92)$$

2.4.3 – Non-linear stiffness matrix $[K_4]$

In this section the non-linear stiffness matrix, $[K_4]$ is derived. The $[K_4]$ matrix is a quadratic function of the transverse displacement w and depends only upon the out-of-plane shape functions and corresponding generalised displacements $\{q_w\}$. Within the three non-linear stiffness matrices, $[K_4]$ is the main source of geometric non-linearity.

From equation (2.64),

$$\begin{aligned} \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T [E] \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega &= \{\delta q\}^T [K_4] \{q\} = \\ &= \int_{\Omega} \left(\delta \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix}^T \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) d\Omega = \int_{\Omega} \left(\delta \{\varepsilon_L^p\}^T [A] \{\varepsilon_L^p\} \right) d\Omega \end{aligned} \quad (2.93)$$

From the definition of $\{\varepsilon_L^p\}$ in equation (2. 51), results that

$$\begin{aligned}
& \int_{\Omega} \delta \left(\{\varepsilon_L^p\}^T [A] \{\varepsilon_L^p\} \right) d\Omega = \\
& = \frac{1}{2} \{\delta q_w\}^T \int_{\Omega} \left[\begin{array}{c} \left[\begin{array}{c} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \\ \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \\ 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \end{array} \right]^T \\ \left[\begin{array}{ccc} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{array} \right] \\ \left[\begin{array}{c} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} \\ \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \end{array} \right] \end{array} \right] d\Omega \\
& = \frac{1}{2} \{\delta q_w\}^T \int_{\Omega} \left[\begin{array}{c} \left[\begin{array}{c} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{11} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{12} + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{16} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{12} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{22} + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{26} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{16} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{26} + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{66} \end{array} \right]^T \\ \times \\ \left[\begin{array}{c} \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} \\ \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} \\ \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \end{array} \right] \end{array} \right] d\Omega \\
& = \frac{1}{2} \{\delta q_w\}^T \int_{\Omega} \left(\left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{11} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{12} + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{16} \right) \times \right. \\
& \times \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T \{q_w\} + \left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{12} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{22} + \right. \\
& \left. \left. + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{26} \right) \times \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T \{q_w\} + \right. \\
& \left. \left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{16} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{26} + 2\{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{66} \right) \times \right. \\
& \left. \times \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \{q_w\} \right) d\Omega
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \{\delta q_w\}^T \int_{\Omega} \left(\left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{11} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{12} + 2 \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{16} \right) \times \right. \\
&\times \frac{1}{2} \{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T + \left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{12} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{22} + \right. \\
&+ 2 \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{26} \left. \right) \times \frac{1}{2} \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T + \\
&\left(\{q_w\}^T \{N_{,x}^w\} \{N_{,x}^w\}^T A_{16} + \{q_w\}^T \{N_{,y}^w\} \{N_{,y}^w\}^T A_{26} + 2 \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T A_{66} \right) \times \\
&\times \{q_w\}^T \{N_{,x}^w\} \{N_{,y}^w\}^T \left. \right) d\Omega \{q_w\} \\
&= \{\delta q\}^T [K_4] \{q\}
\end{aligned}$$

(2. 94)

2.4.4 - Shear Linear Stiffness Matrix $[K_1^y]$

From equation (2. 59), and considering $\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$,

$$\int_{\Omega} \left(\{\delta \gamma\}^T \{Q\} d\Omega \right) = \int_{\Omega} \delta \begin{Bmatrix} \gamma_{zx} \\ \gamma_{yz} \end{Bmatrix}^T \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} d\Omega$$

From $\begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \{N_{,y}^w\}^T & 0 & -\{N^{\Phi_x}\}^T \\ \{N_{,x}^w\}^T & \{N^{\Phi_y}\}^T & 0 \end{bmatrix} \begin{Bmatrix} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{Bmatrix}$, the integral

becomes

$$\begin{aligned}
& \int_{\Omega} \delta \left[\begin{array}{ccc} \{N_{,y}^w\}^T & 0 & -\{N^{\Phi_x}\}^T \\ \{N_{,x}^w\}^T & \{N^{\Phi_y}\}^T & 0 \end{array} \right] \left[\begin{array}{c} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{array} \right] d\Omega = \\
& = \left\{ \begin{array}{c} \delta q_w \\ \delta q_{\Phi_y} \\ \delta q_{\Phi_x} \end{array} \right\}^T \int_{\Omega} \left[\begin{array}{c} \{N_{,y}^w\} \bar{Q}_{44} \{N_{,y}^w\}^T + \{N_{,x}^w\} \bar{Q}_{45} \{N_{,y}^w\}^T + \{N_{,y}^w\} \bar{Q}_{45} \{N_{,x}^w\}^T + \{N_{,x}^w\} \bar{Q}_{55} \{N_{,x}^w\}^T \\ \{N^{\Phi_y}\} \bar{Q}_{45} \{N_{,y}^w\}^T + \{N^{\Phi_y}\} \bar{Q}_{55} \{N_{,x}^w\}^T \\ -\{N^{\Phi_x}\} \bar{Q}_{45} \{N_{,y}^w\}^T - \{N^{\Phi_x}\} \bar{Q}_{55} \{N_{,x}^w\}^T \\ \{N_{,y}^w\} \bar{Q}_{45} \{N^{\Phi_y}\}^T + \{N_{,x}^w\} \bar{Q}_{55} \{N^{\Phi_y}\}^T - \{N_{,y}^w\} \bar{Q}_{44} \{N^{\Phi_x}\}^T - \{N_{,x}^w\} \bar{Q}_{45} \{N^{\Phi_x}\}^T \\ \{N^{\Phi_y}\} \bar{Q}_{55} \{N^{\Phi_y}\}^T - \{N^{\Phi_y}\} \bar{Q}_{45} \{N^{\Phi_x}\}^T \\ -\{N^{\Phi_x}\} \bar{Q}_{55} \{N^{\Phi_y}\}^T \{N^{\Phi_x}\} \bar{Q}_{45} \{N^{\Phi_x}\}^T \end{array} \right] d\Omega \left\{ \begin{array}{c} q_w \\ q_{\Phi_y} \\ q_{\Phi_x} \end{array} \right\} \\
& = \{\delta q\}^T [K_1^\gamma] \{q\} \tag{2.95}
\end{aligned}$$

3. DISPLACEMENT SHAPE FUNCTIONS

As referred in section 2.1, a matrix of shape functions is considered, and four sets of shape functions are required: one set of in-plane shape functions, one set of out-of-plane shape functions, one set of rotation about x shape functions and one set of rotation about y shape functions (2. 10)-(2. 13).

The HFEM relies on the utilization of high order shape functions and ill-conditioning is common in high order polynomials. For this reason, trigonometric shape functions are suggested in [2.7]. Nevertheless, with the Rodrigue's form of Legendre polynomials no ill-condition problems were met and it was decided to continue using them. Thus, the in-plane shape functions are given by

$$g_{r-2} = \sum_{n=0}^{INT(r/2)} \frac{(-1)^n (2r-2n-5)!!}{2^n n!(r-2n-1)!} \xi^{r-2n-1}, r > 2 \tag{2.96}$$

And the out-of-plane shape functions are given by

$$f_r = \sum_{n=0}^{INT(r/2)} \frac{(-1)^n (2r-2n-7)!!}{2^n n!(r-2n-1)!} \xi^{r-2n-1}, r > 4 \quad (2.97)$$

where $r!! = r(r-2)\dots(2 \text{ or } 1)$, $0!! = (-1)!! = 1$ and $INT(r/2)$ denotes the integer part of $r/2$. The rotation shape functions about x and y are equal to the in-plane shape functions.

In APPENDIX A, the first out-of-plane and in-plane shape functions are plotted; the shape functions f_r , $1 \leq r \leq 4$ are cubic polynomials not represented.

It can be seen from the plots that:

- (i) the in-plane shape functions have zero displacements at $\xi = -1$ and $\xi = 1$; the Legendre ($r > 4$) out-of-plane shape functions have both zero displacements and slope at these points. These shape functions satisfy fully clamped boundary conditions;
- (ii) the odd number in-plane and Legendre out-of-plane shape functions are symmetric, while the even number are anti-symmetric. The adequate shape functions regarding the symmetries of the problem under study can be chosen, thus reducing the number of degrees of freedom.

4. NEWMARK METHOD

In the previous sections, a finite element model was derived and the undamped equations of motion were obtained. Introducing damping matrix $[C]$, these equations may be written as:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{P\} \quad (2.98)$$

Where $[M]$ and $[K]$ are the mass and stiffness matrices; $\{P\}$ is the vector of externally applied loads.

The equations of motion are integrated in time domain, directly, that is without transformation of co-ordinates [2.6]. In essence, direct numerical integration is based in two ideas. First, instead of trying to satisfy (2.98) at any time t , it is aimed to satisfy (2.98) only at discrete time intervals Δt apart. The second idea in which a direct integration method is based is that a variation of displacements and accelerations within each time interval Δt is assumed.

In the following, displacement and acceleration vectors at time 0 are denoted by $\{\ddot{q}\}^0$ and $\{\dot{q}\}^0$, respectively, are known, and let the solution to the differential equation (2.98) be required from time 0 to time t_f , in intervals of Δt .

To determine the solution of the displacements and accelerations at time $t + \Delta t$, the equations (2.98) at time $t + \Delta t$ are also considered:

$$[M]\{\ddot{q}\}^{t+\Delta t} + [C]\{\dot{q}\}^{t+\Delta t} + [K]\{q\}^{t+\Delta t} = \{P\}^{t+\Delta t} \quad (2.99)$$

It is assumed that the accelerations within each time interval, Δt are given by

$$\{\ddot{q}\} = \{\ddot{q}\}^t + \frac{1}{\Delta t} \left(\{\ddot{q}\}^{t+\Delta t} - \{\ddot{q}\}^t \right) \tau, \quad 0 \leq \tau \leq \Delta t \quad (2.100)$$

Integrating (2.100) gives

$$\{\dot{q}\} = \{\dot{q}\}^t + \{\ddot{q}\}^t \tau + \frac{1}{2\Delta t} \left(\{\ddot{q}\}^{t+\Delta t} - \{\ddot{q}\}^t \right) \tau^2 \quad (2.101)$$

since $\{\dot{q}\} = \{\dot{q}\}^t$ when $\tau = 0$. Integrating again gives

$$\{q\} = \{q\}^t + \{\dot{q}\}^t \tau + \frac{1}{2} \{\ddot{q}\}^t \tau^2 + \frac{1}{6\Delta t} \left(\{\ddot{q}\}^{t+\Delta t} - \{\ddot{q}\}^t \right) \tau^3 \quad (2.102)$$

since $\{q\} = \{q\}^t$ when $\tau = 0$.

Evaluating (2.101) and (2.102) at $\tau = \Delta t$ gives

$$\{\dot{q}\}^{t+\Delta t} = \{\dot{q}\}^t + \frac{\Delta t}{2} \left(\{\ddot{q}\}^t + \{\ddot{q}\}^{t+\Delta t} \right) \quad (2.103)$$

and

$$\{q\}^{t+\Delta t} = \{q\}^t + \{\dot{q}\}^t \Delta t + \frac{(\Delta t)^2}{6} \left(2\{\ddot{q}\}^t + \{\ddot{q}\}^{t+\Delta t} \right) \quad (2.104)$$

In the Newmark method, equations (2.103) and (2.104) are assumed to take the form [2.6]

$$\{\dot{q}\}^{t+\Delta t} = \{\dot{q}\}^t + \Delta t \left[(1-\alpha)\{\ddot{q}\}^t + \alpha\{\ddot{q}\}^{t+\Delta t} \right] \quad (2.105)$$

and

$$\{q\}^{t+\Delta t} = \{q\}^t + \{\dot{q}\}^t \Delta t + (\Delta t)^2 \left[\left(\frac{1}{2} - \delta \right) \{\ddot{q}\}^t + \delta \{\ddot{q}\}^{t+\Delta t} \right] \quad (2.106)$$

Taking $\alpha = \frac{1}{2}$ and $\delta = \frac{1}{4}$ corresponds to assuming that the acceleration is constant and equal to the average value $\left(\{\ddot{q}\}^t + \{\ddot{q}\}^{t+\Delta t} \right) / 2$ within the interval $(t, t + \Delta t)$.

The response at time $t + \Delta t$ is obtained by evaluating the equation of motion at time $t + \Delta t$, that is

$$[M]\{\ddot{q}\}^{t+\Delta t} + [C]\{\dot{q}\}^{t+\Delta t} + [K]\{q\}^{t+\Delta t} = \{P\}^{t+\Delta t} \quad (2.107)$$

In order to obtain an equation for $\{q\}^{t+\Delta t}$, equation (2.106) is solved for $\{\ddot{q}\}^{t+\Delta t}$ which gives

$$\{\ddot{q}\}^{t+\Delta t} = \frac{1}{\delta(\Delta t)^2} \left(\{q\}^{t+\Delta t} - \{q\}^t \right) - \frac{1}{\delta\Delta t} \{\dot{q}\}^t - \left(\frac{1}{2\delta} - 1 \right) \{\ddot{q}\}^t \quad (2.108)$$

Replacing $\{\ddot{q}\}^{t+\Delta t}$, given in (2.108) in (2.105) gives

$$\{\dot{q}\}^{t+\Delta t} = \frac{\alpha}{\delta\Delta t} \left(\{q\}^{t+\Delta t} - \{q\}^t \right) + \left(1 - \frac{\alpha}{\delta} \right) \{\dot{q}\}^t + \Delta t \left(1 - \frac{\alpha}{2\delta} \right) \{\ddot{q}\}^t \quad (2.109)$$

Substituting (2.108) and (2.109) into (2.107),

$$\begin{aligned} & [M] \left(\frac{1}{\delta(\Delta t)^2} \left(\{q\}^{t+\Delta t} - \{q\}^t \right) - \frac{1}{\delta\Delta t} \{\dot{q}\}^t - \left(\frac{1}{2\delta} - 1 \right) \{\ddot{q}\}^t \right) + \\ & + [C] \left(\frac{\alpha}{\delta\Delta t} \left(\{q\}^{t+\Delta t} - \{q\}^t \right) + \left(1 - \frac{\alpha}{\delta} \right) \{\dot{q}\}^t + \Delta t \left(1 - \frac{\alpha}{2\delta} \right) \{\ddot{q}\}^t \right) + \\ & + [K] \{q\}^{t+\Delta t} = \{P\}^{t+\Delta t} \Leftrightarrow \\ & \Leftrightarrow \frac{1}{\delta(\Delta t)^2} [M] \left(\{q\}^{t+\Delta t} - \{q\}^t \right) - \frac{1}{\delta\Delta t} [M] \{\dot{q}\}^t - \left(\frac{1}{2\delta} - 1 \right) [M] \{\ddot{q}\}^t + \\ & + \frac{\alpha}{\delta\Delta t} [C] \left(\{q\}^{t+\Delta t} - \{q\}^t \right) + \left(1 - \frac{\alpha}{\delta} \right) [C] \{\dot{q}\}^t + \Delta t \left(1 - \frac{\alpha}{2\delta} \right) [C] \{\ddot{q}\}^t + \\ & + [K] \{q\}^{t+\Delta t} = \{P\}^{t+\Delta t} \end{aligned} \quad (2.110)$$

Solving for $\{q\}^{t+\Delta t}$,

$$\begin{aligned} & \left(\frac{1}{\delta(\Delta t)^2} [M] + \frac{\alpha}{\delta(\Delta t)^2} [C] + [K] \right) \{q\}^{t+\Delta t} = \\ & \{P\}^{t+\Delta t} + \frac{1}{\delta(\Delta t)^2} [M] \{q\}^t + \frac{1}{\delta\Delta t} [M] \{\dot{q}\}^t + \left(\frac{1}{2\delta} - 1 \right) [M] \{\ddot{q}\}^t + \\ & + \frac{\alpha}{\delta\Delta t} [C] \{q\}^t - \left(1 - \frac{\alpha}{\delta} \right) [C] \{\dot{q}\}^t - \Delta t \left(1 - \frac{\alpha}{2\delta} \right) [C] \{\ddot{q}\}^t \end{aligned}$$

Placing together terms in $\{\ddot{q}\}^t$, $\{\dot{q}\}^t$ and $\{q\}^t$, in the second member,

$$\begin{aligned}
& \left(\frac{1}{\delta(\Delta t)^2} [M] + \frac{\alpha}{\delta \Delta t} [C] + [K] \right) \{q\}^{t+\Delta t} = \\
& = \{P\}^{t+\Delta t} + \left(\frac{1}{\delta(\Delta t)^2} [M] + \frac{\alpha}{\delta \Delta t} [C] + [K] \right) \{q\}^t + \\
& \left(\frac{1}{\delta \Delta t} [M] - \left(1 - \frac{\alpha}{\delta}\right) [C] \right) \{\dot{q}\}^t + \left(\left(\frac{1}{2\delta} - 1\right) [M] - \Delta t \left(1 - \frac{\alpha}{2\delta}\right) [C] \right) \{\ddot{q}\}^t
\end{aligned} \tag{2.111}$$

If $\{q\}^t$, $\{\dot{q}\}^t$ and $\{\ddot{q}\}^t$ are known, then $\{q\}^{t+\Delta t}$ can be calculated using (2.111).

Equations (2.108) and (2.109) can be used to determine $\{\ddot{q}\}^{t+\Delta t}$ and $\{\dot{q}\}^{t+\Delta t}$. The time history of the response is obtained from time 0 to time t_f , in intervals of Δt .

Therefore, from equations (2.108), (2.109) and (2.111) the Newmark Method is given by

$$\begin{aligned}
& (a_1 [M] + a_2 [C] + [K]) \{q\}^{t+\Delta t} = \{P\}^{t+\Delta t} + (a_1 [M] + a_2 [C]) \{q\}^t + \\
& \quad + (a_3 [M] - a_4 [C]) \{\dot{q}\}^t + (a_5 [M] - a_6 [C]) \{\ddot{q}\}^t \\
& \{\ddot{q}\}^{t+\Delta t} = a_1 (\{q\}^{t+\Delta t} - \{q\}^t) - a_3 \{\dot{q}\}^t - a_5 \{\ddot{q}\}^t \\
& \{\dot{q}\}^{t+\Delta t} = a_2 (\{q\}^{t+\Delta t} - \{q\}^t) + a_4 \{\dot{q}\}^t + a_6 \{\ddot{q}\}^t
\end{aligned} \tag{2.112}$$

where

$$\begin{aligned}
a_1 &= \frac{1}{\delta(\Delta t)^2}, & a_2 &= \frac{\alpha}{\delta \Delta t}, \\
a_3 &= \frac{1}{\delta \Delta t}, & a_4 &= \left(1 - \frac{\alpha}{\delta}\right), \\
a_5 &= \left(\frac{1}{2\delta} - 1\right), & a_6 &= \Delta t \left(1 - \frac{\alpha}{2\delta}\right)
\end{aligned} \tag{2.113}$$

The set of linear equations given in (2.112) is solved in order to determine $\{q\}^{t+\Delta t}$.

In reference [2.6] the stability of the method is investigated and it is stated that the method is unconditionally stable if

$$\alpha \geq 0,50 \text{ and } \delta \geq 0,25(0,5 + \delta)^2 \quad (2.114)$$

Unless α is taken to be $\frac{1}{2}$, the method introduces artificial damping, which can be negative (when $\alpha < \frac{1}{2}$). In the Newmark integration scheme, as an unconditionally stable method (choosing parameters verifying (2.114), the time step Δt is based upon the period corresponding to the highest frequency likely to contribute to the response, ω_n . According to reference [2.6], good accuracy is obtained with a time step given by $\omega_n \Delta t = \pi/50$.

The Newmark method can be extended to non-linear dynamic analysis. This requires that iteration must be performed at each time step in order to satisfy equilibrium. Also, the non-linear stiffness matrix must be formed and triangularized at each iteration or at selective points in time.

5. CLOSING COMMENTS

The mathematical model of the p -version HEFM for asymmetrically laminated rectangular plates is given in this chapter. The equations of motion are obtained and the matrices in the equations of motion are derived for the asymmetrically laminated rectangular plates, with a detailed description of the matrices involved. All these matrices are finally expressed as the integration of shape functions and their derivatives. In the implementation of the model, the differentiation and the integration of the polynomials will be calculated symbolically to find out the exact values via symbolic computation method using *MAPLE*.

The model is derived in time domain by applying the finite element method, the principle of virtual work and the d'Alembert's principle. Because the problems to be analysed do not involve singularities and the geometry of the plate is very regular, only one element was used. The model developed in this chapter is applied to the study of the geometrical non-linear vibrations of plates in following chapters.

